Lecture 28

Szemerédi's Regularity lemma

Roughly speaking, it states that every large graph admits a partition into bounded # parts s.t. between almost all pairs of parts, the induced bipartite subgraphs behave pseudorandomly.

Essence: Approximating large structures by (random blowup) small structures (w/ low complexity)

- Informally, let's first see how one can use regularity lem.

1) Reduce an extremal problem A on large graphs to a problem B on small (weighted) graphs (using the random behavior of the regular partition, counting lem...)

2). Problem B is usually easier to solve.

Example Erdős-Stone-Simonovits:

$$ex(n, H) \le \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\frac{n^2}{2}$$

Reduction is done via counting lem (informal):

- for any graph G, there is a (weighted) graph R on $O(1)$ vxs s.t.

P1) $\forall$ fixed H, the subgraph density of H in R is roughly the same as that in G.

P2) If R contains a homomorphic copy of H then G contains a copy of H.

$$\left(H \subseteq R[s] \Rightarrow H \subseteq G \right)$$

i.e. if G is H-free $\Rightarrow$ then R is $K_{\chi(H)} \in$ Hom(H)-free

all hom. image of H

- Informal pf of ESS using reg. lem.
  - Let $r = \chi(H) - 1$. By P1) w./ $H = K_2$, we just need to bound the edge-density of R from above by $1 - \frac{1}{r}$
  - By P2) R is K_{r+1}-free. The desired bound follows from Turán's thm.

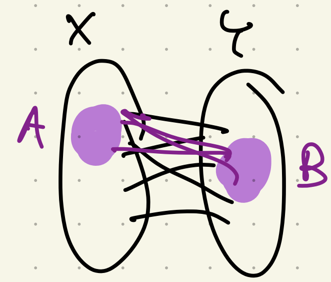
Survey: Komlós - Simonovits

§ Formal Setup.

- basic notion: ε-regular pair, which measures the pseudorandomness / regularity of the induced bip. subgraph between the pair. The parameter $0 < \varepsilon < 1$ is the precision of the regularity; the smaller ε is, the more random like the pair is.

Def [Regular pair] Given $G=(V,E)$ and disjoint vx subsets $X, Y \subseteq V$, the density of the pair (X, Y) is

$$d(X,Y) := \frac{e(X,Y)}{|X||Y|}$$



For $\varepsilon > 0$, the pair (X,Y) is ε-regular

if $\forall A \subseteq X, B \subseteq Y$ w./ $|A| \geq \varepsilon |X|, |B| \geq \varepsilon |Y|$ satisfy

$$|d(A,B) - d(X,Y)| < \varepsilon$$

Additionally, if $d(X,Y) \geq \delta$ for some $\delta > 0$, we say that (X,Y) is (ε, δ)-regular.

- In other words, a regular pair (X,Y) has uniform edge distribution.

Def (Regular partition) A partition $V = V_0 \cup V_1 \cup \dots \cup V_r$ is ε-regular, if

(i) $|V_0| \leq \varepsilon |V|$; (exceptional set)

(ii) $|V_1| = \cdots = |V_r|$;

(iii) all but εr^2 pairs (V_i, V_j) w./ $1 \leq i < j \leq r$ are ε-regular.

Rmk: • We do not assume V_i is larger than V_0.

• In the defn. of reg. partition, we can also have no exceptional set V_0 and instead have $||V_i|-|V_j|| \le 1$

• Degree version: we can require that $\forall\ i \in [r]$, all but εr pairs involving V_i are ε-regular.

Szemerédi Reg. Lem. 1976

Given $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $M = M(\varepsilon, m)$, s.t. any graph G admits an ε-regular partition $V = V_0 \cup V_1 \cup \dots \cup V_r$ w./ $m \le r \le M$.

Rmk: • We usually think of ε in the reg. lem. as a very small const. i.e. $o(1)$.

• Both the lower & upper bounds $m \le r \le M$ on the # parts of the partition are meaningful.

- If there is no lower bound, then the trivial partition $V = V$, then it is vacuously a reg. partition but useless.

- We usually want the proportion of edges inside V_i, $i \in [r]$ to be negligible, which requires r to be large.

- Upp. bd. is needed as in the pf of counting lem. relies crucially on the fact that r is bounded.

- If G does not have positive edge-density, then the reg. lem. does not say much about G.

- The εr^2 exceptional irregular pairs are needed. b/c of the following example:

  Half graph $G = (A \cup B, E)$ where $A = B = [n]$

  put $ab \in E(G) \iff a \geq b$.

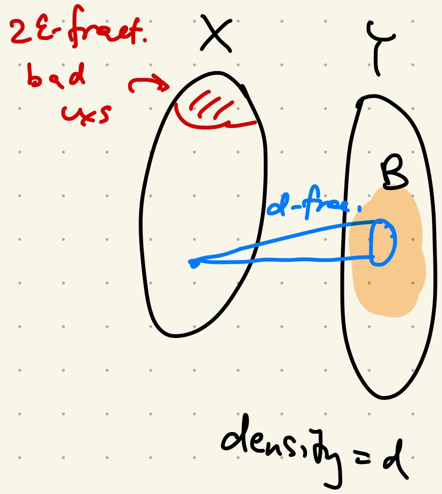
  - $d(A, B) = 1/2$
  - There are εr irregular pairs in any partition

- The upper bound on the # parts M is rather large: tower of 2s w/ height $2\varepsilon^{-5}$.

  Gowers gave a construction showing that a tower of 2s w/ height $\varepsilon^{-1/16}$ is needed.

- The following states that between a reg. dense pair, almost every vertex has the "correct" degree to any large subset of the other side.

Lem Let (X,Y) be an ε-regular pair w/ density d, and $B \subseteq Y$ w/ $|B| \geq \varepsilon |Y|$, then all but $2\varepsilon|X|$ vxs in X have degree $(d \pm \varepsilon)|B|$ in B.



Pf: Let $A \subseteq X$ be the set of vxs w/ "small" deg in B, i.e.

$$d(v,B) < (d-\varepsilon)|B|.$$

Suppose $|A| > \varepsilon |X|$.

$$\Rightarrow \quad d(A,B) = \frac{e(A,B)}{|A|\cdot|B|} < \frac{|A|\cdot(d-\varepsilon)|B|}{|A|\cdot|B|} = d-\varepsilon.$$

Contradicting (X,Y) being ε-reg. $\Rightarrow$ $|A| \leq \varepsilon |X|$.

- Similarly, the same bd holds for the set of vxs of "large deg" to B, i.e. $d(v,B) > (d+\varepsilon)|B|$.

- Given a regular pair (X,Y), one can show that almost all pairs from one part have the "correct" codegree to large subsets of the other side.

Exer: Formulate the above codeg. statement and prove it.

The next lem states that regularity is inherited by large subsets of pairs (w./ slightly worse regularity). It allows us to refine a regular partition to get additional properties without losing regularity.

Lem. (Slicing lem) Let $V = V_0 \cup V_1 \cup \cdots \cup V_r$ be an ε-regular partition. Further refine each part into s parts: $V_i = V_i^1 \cup \cdots \cup V_i^s$. The new partition (w./ $sr+1$ parts) is $O(s\varepsilon)$-regular.

Rmk. $O(s\varepsilon)$-reg. implies implicitly that in the slicing lem. $s \ll 1/\varepsilon$

Exer: Prove the slicing lem.