Lecture 27

Goal $S(n) \leq T(n) \cdot 2^{O\left(\frac{n^{3/4}}{\log^{3/2} n}\right)}$

Idea (Dyatic) partition and utilise asymm. Turán # for C_4.

Pf: - Let $S \subseteq [n]$ be a multi. Sidon set

We may assume S has no perfect squares, as there are $\leq \sqrt{n}$ perfect squares in $[n]$, contributing a negligible $2^{\sqrt{n}}$ factor.

- Partition $S = A \cup B$, where

$A = \{a \in S : \exists\, n^{2/3} < p \text{ prime s.t. } p \mid a\}$

$B = S \setminus A = \{a \in S : \text{all prime divisors of } a \text{ are } \leq n^{2/3}\}$

- Further partition $A = A_1 \cup A_2$, where

$A_1 = \{a \in A : \exists\, n^{2/3} < p \text{ prime s.t. } p \mid a \text{ but } p \nmid b \text{ for any } b \in S \setminus \{a\}\}$

$A_2 = A \setminus A_1 = \{a \in A : \exists\, n^{2/3} < p \text{ prime and } \exists\, b \in S \setminus \{a\} \text{ s.t. } p \mid (a,b)\}$

- By Lem 1, we can write each $a \in B$ as $a = u \cdot v$ w/ $v \leq u \leq n^{2/3}$.

We fix one such representation u, v s.t. v is minimum.

$(u,v) = (u_a, v_a) : uv = a,\ v \leq u \leq n^{2/3}$

Lem 1 $\forall a \in [n]$, we can write it as $a = u \cdot v$ w/ $v \leq u$ s.t.

- either u is a prime $\geq n^{2/3}$
- or $v \leq u \leq n^{2/3}$.

Exer.

$\forall\, u'v' = a \implies v \le \min\{u', v'\}$

- Further partition $B = B_1 \cup B_2 \cup B_3$.
  - $B_1 = \{a \in B : v \le n^{1/3}\}$
  - $B_2 = \{a \in B : n^{1/3} \le v \le \frac{n^{1/2}}{(\log n)^8}\}$
  - $B_3 = \{a \in B : \frac{n^{1/2}}{(\log n)^8} \le v \le n^{1/2}\}$

- $S = \underbrace{A_1}_{\text{leading term } T(n)} \cup \underbrace{A_2 \cup B_1 \cup B_2}_{\text{negligible b/c they are small}} \cup \underbrace{B_3}_{\text{first lower order term } 2^{\Theta\left(\frac{n^{3/4}}{\log^{3/2} n}\right)}}$

$|A_2|, |B_1|, |B_2| \le \frac{8n^{3/4}}{(\log n)^4}$.

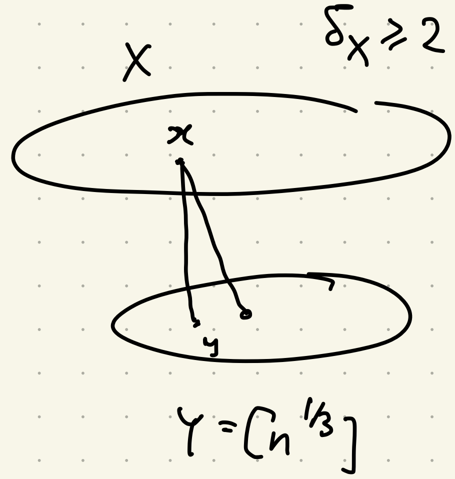
such small sets $\le \sum_{i \le \frac{8n^{3/4}}{(\log n)^4}} \binom{n}{i} \le 2^{\frac{12 n^{3/4}}{(\log n)^3}}$

- A_1: elements in A_1 has a prime divisor $p > n^{2/3}$ and p does not divide any others. $\Rightarrow \forall$ prime $p > n^{2/3}$,

  A_1 contains ≤ 1 multiple of p

  $\Rightarrow$ # choices for $A_1 \le \prod_{\substack{p > n^{2/3} \\ \text{prime}}} \left(\lfloor \frac{n}{p} \rfloor + 1\right) = T(n)$

- A_2: Consider primes $n^{2/3} < p \leq n$ that divides ≥ 2 elements of S. For each such prime p, let $m_p \geq 2$ be the # multiples of p in S.



Aux. bip. Γ on sets X and Y

- X contains all primes $p \in (n^{2/3}, n]$ w/ $m_p \geq 2$
- $Y = [n^{1/3}]$
- $x \sim y$ iff $x \cdot y \in S$

- Note that $\forall x \in X$ $d(x) = m_x \geq 2$.
- S multi. Sidon $\Rightarrow$ Γ is C_4-free
- $|A_2| = e(\Gamma)$

Thm 2 $\Rightarrow$ $e(\Gamma) \leq \pi(n) \approx \frac{n}{\log n}$

$\boxed{\text{Thm 2. } \forall\, m \leq n, \; ex(m, n, C_4) \leq m n^{1/2} + n}$

Using min. deg from X side is ≥ 2, we can do better

A hat in Γ is a P_3 w/ mid-pt in X.

Γ is C_4-free $\Rightarrow$ no two hats sharing the same pair of endpts in Y

i.e. $$\sum_{p \in X} \binom{m_p}{2} \leq \binom{|Y|}{2} = \binom{n^{1/3}}{2}$$

$$\Rightarrow \quad |A_2| = e(\Gamma) = \sum_{p \in X} m_p \overset{\delta_x \geq 2}{\leq} 2 \sum_{p \in X} \binom{m_p}{2} \leq n^{2/3}.$$

$\forall p: \ m_p \leq 2\binom{m_p}{2}$

- B_1: By defn. $\forall a \in B_1$, $a = uv$ satisfies $v \leq n^{1/3}$, $u \leq n^{2/3}$

Aux. bip. Γ on $U \cup V$, $U = [n^{2/3}]$, $V = [n^{1/3}]$

$u \in U, v \in V$ $u \sim v \iff uv = a$ is the chosen representation of a.

Similarly Γ is C_4-free as B_1 is multi. Sidon

By Thm 2 $\Rightarrow |B_1| = e(\Gamma) \leq |V| \cdot |U|^{1/2} + |U| = 2n^{2/3}$.

- B_2: Let $R = 8 \log_2 \log n$, dyatic partition B_2 into $B_2^1, B_2^2, \ldots$

s.t. $\forall r \geq 1$, $\quad B_2^r = \left\{ a \in B_2 : \dfrac{n^{1/2}}{2^{R+r}} < v \leq \dfrac{n^{1/2}}{2^{R+r-1}} =: M_r \right\}$

- As $v > n^{1/3}$, there are $\leq \log n$ subsets B_2^r.
- $\forall a = uv \in B_2^r$

$$u \leq 2^{R+r} \cdot n^{1/2} := N_r \leq n^{2/3}$$

Build Aux. bip. Γ^r for B_2^r. on sets $U = [N_r]$, $V = [M_r]$

$u \sim v$ iff $uv = a$ is the chosen representation.

- Γ^r C_4-free $\Rightarrow$ $|B_2^r| = e(\Gamma^r) \underset{\text{Thm 2}}{\leq} N_r + \sqrt{N_r} \cdot M_r$

$$\leq \dots \leq n^{2/3} + \frac{2n^{3/4}}{(\log n)^4} \cdot \frac{1}{2^{r/2}}$$

$$\Rightarrow |B_2| = \sum_{r \leq \log n} |B_2^r| \leq \frac{8n^{3/4}}{(\log n)^4}.$$

- B_3: Let $R = 8\log_2 \log n$, partition $B_3 = B_3^1 \cup B_3^2 \cup \dots$

where $\forall r \in [R]$:

$$B_3^r = \left\{ a \in B_3 : \frac{n^{1/2}}{2^r} < v \leq \frac{n^{1/2}}{2^{r-1}} \right\}$$

- Fix $r \in [R]$ and $a \in B_3^r$ w/ repr. $a = u \cdot v$.

<u>Claim</u> v has no prime divisor $< n^{1/7}$.

<u>Pf</u>: Suppose $p < n^{1/7}$ is a prime divisor of v.

$$\Rightarrow u \cdot p < 2^r \cdot n^{1/2} \cdot n^{1/7} < n^{2/3}.$$

$$\Rightarrow (u \cdot p, v/p) \text{ is a repr. } \lightning \text{ minimality of } v.$$

By the claim,

$$\text{\# choices for } v \underset{\text{Lem 2}}{\leq} \frac{n^{1/2}}{2^{r-1}} \cdot c \cdot \prod_{\substack{p \text{ prime} \\ p < n^{1/7}}} \left(1 - \frac{1}{p}\right)$$

$$\underset{\text{Lem 3}}{\leq} O\left(\frac{n^{1/2}}{2^r \log n}\right) =: M_r.$$

Lem 2 $\exists c > 0$ s.t. $\forall$ primes $p_1 \leq \dots \leq p_k \leq n$, # integers $m \leq n$ which are not divisible by any of p_i is

$$\leq cn \cdot \prod_{i \in [k]} \left(1 - \frac{1}{p_i}\right)$$

Lem 3 (Merten's estimate)

$$\prod_{\substack{p:\text{ primes}\\ p<n}} \left(1-\frac{1}{p}\right) = \Theta\left(\frac{1}{\log n}\right)$$

<u>Claim</u>. u has not prime divisor between 2^{2r} and $n^{1/7}$.

<u>Pf</u>: Suppose $\exists$ such $p \mid u$

$$\Rightarrow \frac{u}{p} \le \frac{2^r \cdot n^{1/2}}{2^{2r}} \le \frac{n^{1/2}}{2^r} < v$$

and $v \cdot p \le n^{1/2} n^{1/7} \le n^{2/3}$. So $(v \cdot p, u/p)$ ↯ minimality of u ☺

$$\Rightarrow \# \text{ choices for } u \le 2^r \cdot n^{1/2} \cdot c \cdot \prod_{\substack{\text{prime}\\ 2^{2r}<p<n^{1/7}}} \left(1-\frac{1}{p}\right)$$

$$\le O\left(2^r \cdot n^{1/2} \cdot \frac{r}{\log n}\right) := N_r.$$

- Associate B_3^r w./ aux. bip. Γ^r on $U = [N_r]$, $V = [M_r]$

$u \sim v \iff (u,v)$ is the chosen repr. for some $a \in B_3^r$

Lem (*) $\Rightarrow$ # choices for B_3^r

$= \#\ (M_r, N_r)$-vx bip. C_4-free Γ^r.

$$\le 2^{O(M_r N_r^{1/2})},$$

<u>Lem (*)</u> Given $n^{1/2} \log^5 n \le m \le n$

C_4-free bip. w/ partite sets of size m and n is

$$\le 2^{O(mn^{1/2})}$$

where $O(M_r \cdot N_r^{1/2}) = \cdots = O\left(\frac{r^{1/2}}{2^{r/2}} \cdot \frac{n^{3/4}}{(\log n)^{3/2}}\right)$

converge

$\Rightarrow$ # choices for B_3

$$\leq 2^{O\left(\sum_{r \in [R]} M_r \cdot N_r^{1/2}\right)} = 2^{O\left(\frac{n^{3/4}}{\log^{3/2} n}\right)}$$