Lecture 26

Goal

- Subdivision conj $\Longrightarrow$ Rational Turán exponent conj

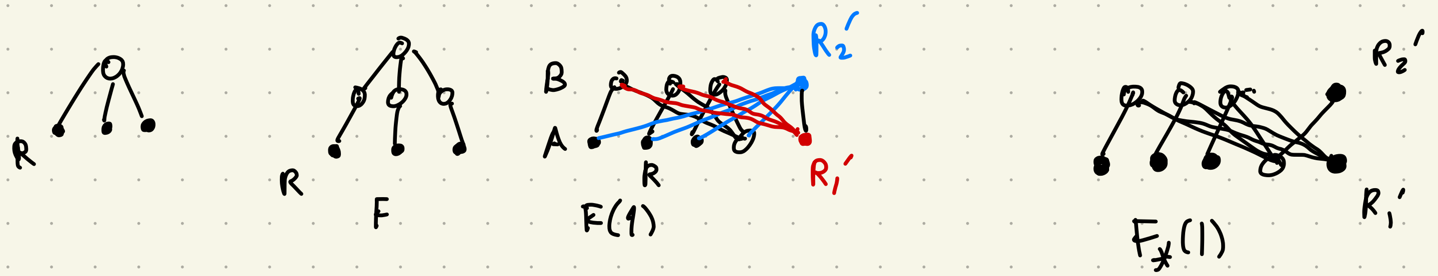
3 ingredients
1) $ex(n, F^{\ell}) = \Omega_{\ell}(n^{2-1/\rho(F)})$
2) Densification lem (reduction Erdős-Simonovits)
3) Sparsification lem (subdivision conj)

- For $t \in \mathbb{N}$, connected bip. F w/ the unique bip. (A, B).

  - $F(t)$: - taking two disjoint t-sets R_1' and R_2' disj from $V(F)$
    - join A and R_2' completely
    - B — " — R_1' — " —
    - R_1' — " — R_2' — " —

  - If F is rooted at R, then consider $F(t)$ as rooted on $R \cup R_1' \cup R_2'$ and let $F_*(t)$ be the rooted graph obtained from $F(t)$ by removing all edges in $R \cup R_1' \cup R_2'$

Exam $F = sub(S_3)$, $t = 1$



Lem (E-Sim. reduction) $t \in \mathbb{N}$, F connected, bip w/ $ex(n, F) = O(n^{2-\alpha})$

$$\Longrightarrow \quad ex(n, F(t)) = O(n^{2-\beta}), \quad \beta^{-1} = \alpha^{-1} + t.$$

Prop $t \in \mathbb{N}$, F balanced, rooted bip.

$\Longrightarrow$ $F(t)$ and $F_*(t)$ are balanced rooted bip.

w/ $\rho(F(t)) = \rho(F_*(t)) = \rho(F) + t$

Def: A rational $r \in [1,2)$ is balancedly realisable by a graph F if $\exists$ a balanced connected rooted bip. F and $\ell_0 \in \mathbb{N}$ s.t. $\rho(F) = \frac{1}{2-r}$ and $\forall \ell \geq \ell_0$, $\Rightarrow$ $ex(n, F^\ell) = \Theta(n^r)$

exercise

Lem (Densification) For $a, b \in \mathbb{N}$ w/ $b > a$, if $2 - \frac{a}{b}$ is balancedly realisable by a graph F, then $2 - \frac{a}{a+b}$ is also balancedly realisable by $F_*(1)$.

Prop. Given a balanced bip. F rooted on an indep R w/ $\rho(F) \geq 1$, $\Rightarrow$ 1-subd. sub(F) is also a balanced rooted bip graph.

- $\mathcal{F}_0$ minimal collection of balanced connected rooted bip. graphs satisfying
  - $\mathcal{F}_0$ includes all stars rooted on the leaves
  - $\mathcal{F}_0$ is closed under taking 1-subd.

    i.e. if $F \in \mathcal{F}_0 \Rightarrow \text{sub}(F) \in \mathcal{F}_0$
  - If $F \in \mathcal{F}_0 \Rightarrow F_*(1) \in \mathcal{F}_0$.

Lem (Sparsification) Suppose for any $F \in \mathcal{F}_0$, $\exists\, \ell_0 = \ell_0(F)$ s.t. Subd. conj holds for F^ℓ for all $\ell \geq \ell_0$. If $a, b \in \mathbb{N}$, $b > a$, are such that $2 - a/b$ is balancedly realisable by a graph $F \in \mathcal{F}_0 \implies 2 - \frac{a+b}{2b}$ is also balancedly realisable by $\mathrm{sub}(F)$.

The following thm shows that Subd. conj $\Rightarrow$ R.T.E. conj.

<u>Thm</u> Suppose that $\forall F \in \mathcal{F}_0$, $\exists \ell_0 = \ell_0(F)$ s.t. Subd. conj holds for F^ℓ for all $\ell \geq \ell_0$ $\implies$ R.T.E. conj holds.

<u>Pf</u>: • Suffices to show that $\forall$ $a, b \in \mathbb{N}$ w./ $a < b$, $2 - a/b$ is balancedly realisable.

• Use induction on $a+b$

By I.H. w.l.o.g. we may a, b are coprime

(1) $a = 1$, $\therefore$ $2 - a/b$ is balancedly realisable by complete bip. graph which is a power of a star rooted at leaves

(2) $2 \leq a < b \leq 2a$. $(a,b)=1 \Rightarrow b < 2a$. Let $a' = 2a - b$, $b' = b$

Note that $a' + b' = 2a < a + b$

• I.H. $\implies$ $2 - \frac{a'}{b'} = 2 - \frac{2a-b}{b}$ is b.r. (balancedly realisable) by some graph in $\mathcal{F}_0$

• Sparsification lem $\implies$ $2 - \frac{2a-b+b}{2b} = 2 - \frac{a}{b}$ is b.r.

(3) $b > 2a$. Let $a'' = a$, $b'' = b - a$

$a'' + b'' = b < a + b$

So I.H. $\implies$ $2 - \frac{a''}{b''} = 2 - \frac{a}{b-a}$ is b.r.

Densification lem $\Rightarrow$ $2 - \frac{a}{b-a+a} = 2 - \frac{a}{b}$ is also b.r.

<u>Pf</u> (Sparsification lem).

- By assumption, $\exists$ balanced. conn. rooted bip. $F \in \mathcal{F}_0$ and l_0 s.t. $\rho(F) = b/a$ and $ex(n, F^l) = \Theta(n^{2-a/b})$ $\forall l \geq l_0$.
$$= \Theta(n^{1+\frac{b-a}{b}})$$

- As $sub(F) \in \mathcal{F}_0$ is also balanced w/ $\rho(sub(F)) = \frac{2b}{a+b}$.

$\Rightarrow$ [B-C] $\Rightarrow$ $\exists l_1 \in \mathbb{N}$ s.t. $\forall l \geq l_1$
$$ex(n, (sub(F))^l) = \Omega\left(n^{2-\frac{1}{\rho(sub(F))}}\right) = \Omega\left(n^{2-\frac{a+b}{2b}}\right)$$

- On the other hand, by assumption, Subd. conj holds for F^l for all $l \geq l_0$
$$\Rightarrow \quad ex(n, sub(F^l)) = O\left(n^{1+\frac{b-a}{2b}}\right) = O\left(n^{2-\frac{a+b}{2b}}\right)$$

- By minimality of $\mathcal{F}_0$, root of all $F \in \mathcal{F}_0$ are indep. sets
$$\Rightarrow \quad sub(F^l) = (sub(F))^l.$$

- Conlon-Janzer: Exponent near 2. ref.

§ Multiplicative Sidon set and C_4-free graphs.

<u>Def</u>: $S \subseteq \mathbb{N}$ multiplicative Sidon set if the products of all pairs in S are distinct. In other words, S does not contain distinct elements satisfying the equation $a_1 a_2 = b_1 b_2$.

Write $s(n)$ = max. size multi. Sidon set in $[n]$.

- Erdős - 1938 $\quad s(n) \geq \pi(n) + c' \dfrac{n^{3/4}}{(\log n)^{3/2}}$

  - 1969 $\quad s(n) \leq \pi(n) + c \dfrac{n^{3/4}}{(\log n)^{3/2}}$

- $\pi(n)$ = prime counting function = # primes in $[n]$

P.N.T. $\Rightarrow$ $\quad \pi(n) = (1 + o(1)) \dfrac{n}{\log n}$.

$$s(n) = \pi(n) + \Theta\left(\frac{n^{3/4}}{\log^{3/2}}\right).$$

<u>Counting?</u>

<u>Q</u>: How many multi. Sidon sets in $[n]$?

[L.- Pach]

<u>Def</u>: $S(n)$ = # of

[Cameron - Erdős] 80s : determined asymp. log. of $S(n)$.

<u>Thm</u> [L.–Pach] $\exists\, C>0$ s.t. # multi Sidon sets in $[n]$ satisfies

$$T(n)\cdot 2^{(\sqrt{2}+o(1))\frac{n^{3/4}}{(\log n)^{3/2}}} \le S(n) \le T(n)\cdot 2^{C\frac{n^{3/4}}{(\log n)^{3/2}}}$$

where $$T(n) := \prod_{p \text{ primes}:\, n^{2/3}\le p\le n} \left(\lfloor n/p \rfloor + 1\right)$$

<u>Rmk</u>. 1) rare example counting result where we have the correct order of mag. of the lower order term.

2) $$T(n) = 2^{O(n^{2/3})} \cdot \prod_{i=1}^{n^{1/3}} \left(1+ 1/i\right)^{\pi(n/i)} = \left(2^{\alpha+o(1)}\right)^{\pi(n)}$$

where $$\alpha = \sum_{i=1}^{\infty} \frac{1}{i} \log_2 \left(1+1/i\right) \approx 1.8146.$$

Lower bound construction.

We shall constr. multi. Sidon sets consisting two parts

A and B.

- $\forall\, a\in A$ has a prime divisor $> n^{2/3}$.
- $\forall\, b\in B$ is a product of two primes $\le n^{1/2}$.

Consider a C_4-free graph G of maximum size on the vertex set

$$V(G) = \{p : p\le n^{1/2} \text{ and } p \text{ is a prime}\}$$

P.N.T. $\Rightarrow$ $$|V(G)| = (2+o(1))\frac{n^{1/2}}{\log n}.$$

$$e(G) = \left(\tfrac{1}{2} + o(1)\right) |V(G)|^{3/2}$$
$$= \left(\sqrt{2} + o(1)\right) \frac{n^{3/4}}{(\log n)^{3/2}}$$

$$\boxed{\begin{array}{l} ex(n, C_4) = \left(\tfrac{1}{2} + o(1)\right) n^{3/2} \\ \text{Reiman} \end{array}}$$

Let $B^* \subseteq [n]$ contain exactly those products pq for $pq \in E(G)$: $B^* = \{ pq : pq \in E(G) \}$.

Note B^* is multi. Sidon and $|B^*| = e(G)$

- If not, B^* contains a sol$^{\underline{n}}$ $(p_1 q_1)(p_2 q_2) = (p_3 q_3)(p_4 q_4)$ w/ distinct $p_i q_i$, $i \in [4]$

But p_i, q_i are primes $\Rightarrow$ the sets $\{p_1, q_1, p_2, q_2\}$ and $\{p_3, q_3, p_4, q_4\}$ are identical.

$\Rightarrow$ C_4 in G $\lightning$

- Observe that for $A \subseteq [n]$, if each $a \in A$ has a unique prime divisor p which does not divide any other element of $A \cup B^*$ $\Rightarrow$ $A \cup B^*$ is multi Sidon.
  - To construct such a set A.

  $\forall$ prime $p > n^{2/3}$, include ≤ 1 multiple of p to A. $\rightsquigarrow$ # choices $= \lfloor n/p \rfloor + 1$

$\Rightarrow$ # choices for A is $\displaystyle\prod_{p \text{ prime} > n^{2/3}} \left(\lfloor n/p \rfloor + 1 \right) = T(n)$

- $\forall B \subseteq B^*$, $A \cup B$ multi. Sidon

$S(n) \geq$ # choices for $A \cup B \geq T(n) \cdot 2^{|B^*|} = T(n) \cdot 2^{(\sqrt{2}+o(1))\frac{n^{3/4}}{(\log n)^{3/2}}}$

as desired.