Lecture 25

- Upper bound. **Rmk:** balancedness is not needed.

Lem. Let (T, R) be a rooted tree w/ at least one root,

$$\Rightarrow \quad ex(n, T^p) = O_p(n^{2 - 1/\rho_T}).$$

$$\rho_T = \frac{e(T)}{|T| - |R|}$$

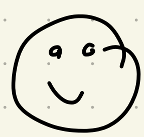
Pf: • Let $t = |T|$ and G be an n-vx graph w/ cn^{2-a} edges where $a = 1/\rho_T$ and $c > 2(t+p)$.

- $G \rightsquigarrow H \subseteq G$ w/ $\delta(H) \geq d(G)/2 = cn^{1-a}$.
  $|H| = h \leq n$
- greedy embedding $\Rightarrow$ # labeled copies of (unrooted) T
  $$\geq h \cdot \delta(H) \cdot (\delta(H) - 1) \cdots (\delta(H) - t + 2) \geq \left(\tfrac{c}{2}\right)^{t-1} h \cdot n^{(t-1)(1-a)}.$$
- # roots $R \leq h^{|R|}$

Averaging $\Rightarrow$ $\exists$ a choice R' of roots in at least

$$\frac{(c/2)^{t-1}\, h \cdot n^{(t-1)(1-a)}}{h^{|R|}} \geq \frac{(c/2)^{t-1}\, n^{(t-1)(1-a)}}{n^{|R|-1}} = \left(\frac{c}{2}\right)^{t-1} \geq p.$$

as $a = 1/\rho_T = \dfrac{t - |R|}{t-1}$.



Lower bound requires the balancedness.

Exer: Give an example of an unbalanced rooted tree (T, R), for which $ex(n, T^p) = \Omega_p(n^{2 - 1/\rho_T})$ is not true.

Lem $\forall$ balanced rooted tree (T,R), $\exists\, p \in \mathbb{N}$ s.t.

$$ex(n, \mathcal{T}^p) = \Omega_p(n^{2-1/\rho_T}).$$

To show that any rational $r \in (1,2)$ can be an exponent for a finite fam. of forbidden graphs, it suffices to find balanced rooted tree (T,R) w/ $2 - 1/\rho_T = r$.

Write $r = 2 - a/b$ for some $a, b \in \mathbb{N}$.

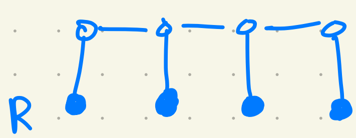
Exam. Take $a, b \in \mathbb{N}$ w/ $a - 1 \leq b < 2a - 1$, let $i = b - a$.

- $T_{a,b}$ rooted tree :
  - Take an unrooted path on $[a]$
  - add an additional rooted leaf to each of the $i+1$ vxs
    $$1,\ \lfloor 1 + a/i \rfloor,\ \lfloor 1 + 2\cdot a/i \rfloor,\ \dots,\ \lfloor 1 + (i-1)\, a/i \rfloor,\ a.$$
- For $b \geq 2a-1$, define $T_{a,b}$ recursively by attaching a rooted leaf to each unrooted vx of $T_{a,b-a}$.

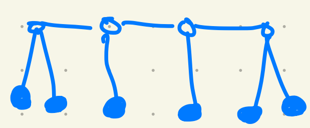
By constr, $T_{a,b}$ has a unrooted vxs and b edges

$$\Rightarrow \rho_{T_{a,b}} = b/a$$

Ex



$T_{4,7}$



$T_{4,9}$

Lem $\forall$ balanced rooted tree (T,R), $\exists\, p \in \mathbb{N}$ s.t.

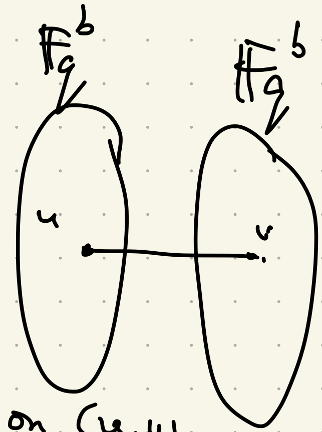
$$\mathrm{ex}(n, \mathcal{T}^p) = \Omega_p\left(n^{2 - 1/\rho_T}\right).$$

Pf (Sketch). • Let (T,R) be a rooted tree w/
- a unrooted vxs
- $r = |R|$ rooted vxs
- b edges.

• $s = 2br$, $d = sb$, $n = q^b$ for some large prime power q.

• Take $2b$-variate indep. unif. random polyn.

$$f_1, \dots, f_a : \mathbb{F}_q^b \times \mathbb{F}_q^b \longrightarrow \mathbb{F}_q,$$ each of deg. at most d.



• (n,n)-vx bip. G w/ each partite set $\cong \mathbb{F}_q^b$ where $uv \in E(G)$ iff all f_i, $i \in [a]$, vanishes on (u,v), i.e.

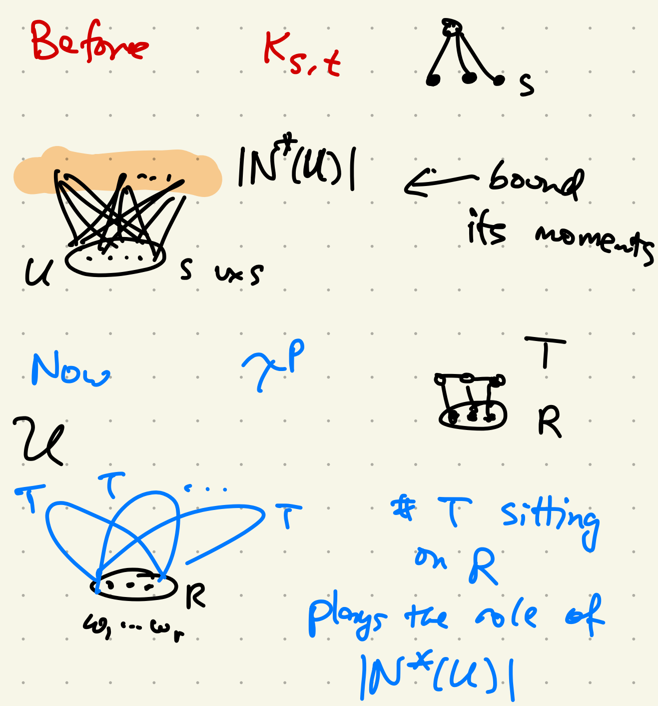
$$f_1(u,v) = \cdots = f_a(u,v) = 0$$

• In expectation, edge density $= \underbrace{q^{-a}}_{= n^{-a/b}} \Rightarrow \mathbb{E}(e(G)) = n^{2 - a/b}$

We are left to show that the expected # copies of graphs in $\mathcal{T}^p$ is negligible.

• Fix vxs $w_1, \dots, w_r$ in G and let $\mathcal{U}$ be the collection of copies of T in G rooted at $\{w_1, \dots, w_r\}$.

We need to bound the moments of $|\mathcal{U}|$.



Claim $\mathbb{E}(|\mathcal{U}|^s) = O_s(1)$. (balancedness used here) $\uparrow$ need to bound its moments.

- Using Lang-Weil bound $\Rightarrow$

  either $|\mathcal{U}| \leq C$ (depending only on T)

  or $|\mathcal{U}| \geq q/2$

- Markov ineq $\Rightarrow$ $\Pr(|\mathcal{U}| > C) = \Pr(|\mathcal{U}| \geq q/2) = \Pr(|\mathcal{U}|^s \geq (q/2)^s)$

$$\leq \frac{O_s(1)}{(q/2)^s}.$$

- Consequently, the expected # $\overset{B}{\overset{\shortparallel}{\text{bad}}}$ $\{\omega_1, \dots, \omega_r\}$ (i.e. sitting in more than C copies of T as roots)

  is $\leq 2n^r \cdot \dfrac{O_s(1)}{(q/2)^s} = o(1)$.

$$\Rightarrow \mathbb{E}\big(e(G) - B\cdot n\big) \geq \mathbb{E}(e(G))/2$$

..... deletion .... $p = C+1$

Pf of Claim: • Notice that $|\mathcal{U}|^s$ counts # ordered collection of s copies of T in G w/ roots $\{\omega_1, \dots, \omega_r\}$.

- Each one of such s-tuple corresponds to a graph H in $\mathcal{T}^s$.

locally indep. (1)

For $H \in \mathcal{T}^s$, the probability that H appears in G

is $q^{-a\cdot e(H)}$

write $N_s(H)$ = # ordered collections of s copies of T w/ roots $\{\omega_1,\dots,\omega_r\}$ whose union is a copy of H

$$\Rightarrow \mathbb{E}(|\mathcal{U}|^s) = \sum_{H\in\mathcal{T}^s} N_s(H)\cdot q^{-a\cdot e(H)}$$

$N_s(H) = O_s(n^{|H|-|R|}) \Rightarrow$

$$= \sum_{H\in\mathcal{T}^s} O_s(n^{|H|-|R|})\cdot q^{-a\cdot e(H)}$$

$$= O_s\Big(\sum_{H\in\mathcal{T}^s} q^{b(|H|-|R|)-a\cdot e(H)}\Big) \overset{\text{Lem}}{=} O_s(1).$$

Lem. • (T,R) balanced rooted tree

• $H\in\mathcal{T}^s$

$\Rightarrow \rho_T \le \dfrac{e(H)}{|H|-|R|}$

Induct on s

Conj (subdivision conj Kang-Kim-L.)

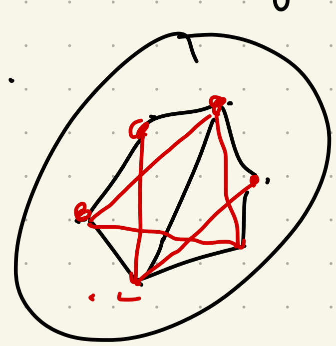
Let F be a bip. graph. If $ex(n,F) = O(n^{1+\alpha})$ for some $\alpha>0$, then

$$ex(n, sub(F)) = O(n^{1+\frac{\alpha}{2}}).$$

Rmk: If an n-vx G has $d(G)\ge cn^{\alpha/2}$

$\Rightarrow$ its square has ave. deg $\gtrsim (d(G))^2$

G



$\Rightarrow$ the square contains F.

While being interesting on its own, in fact we shall see that the Subdivision Conj $\Longrightarrow$ Rational exponent conj.

<u>Known</u>: 1. even cycles, Θ-graphs [Bondy - Sim]

2. complete bip. graph

[Conlon - Lee - Janzer] $ex(n, sub(K_{s,t})) = \Theta(n^{\frac{3}{2} - \frac{1}{2s}})$

$t \gg s$.

<u>Problem</u>: Find more families for which the subd. conj holds.

- We can extend the concept of powers of balanced rooted trees to bip. as follows.

  - (F, R), R roots, F bip.
  - $\underbrace{\forall}_{\text{non-empty}} S \subseteq V(F)$, let $\rho_F(S) := \dfrac{e_S}{|S|}$, e_S = # edges in F incident to a vx in S.
  - let $\rho(F) = \rho_F(V(F) \setminus R)$.

Say (F, R) is balanced if $\rho_F(S) \geq \rho(F)$ $\forall$ nonempty $S \subseteq V(F) \setminus R$.

And define its powers analogously.

Lem (Bukh - Conlon) $\forall$ balanced bip. rooted graph F

w./ $\rho(F) > 0$, $\exists\ \ell_0 = \ell_0(F)$ s.t. $\forall\ \ell \geq \ell_0$

$$ex(n, F^{\ell}) = \Omega_{\ell}\left(n^{2 - 1/\rho(F)}\right).$$