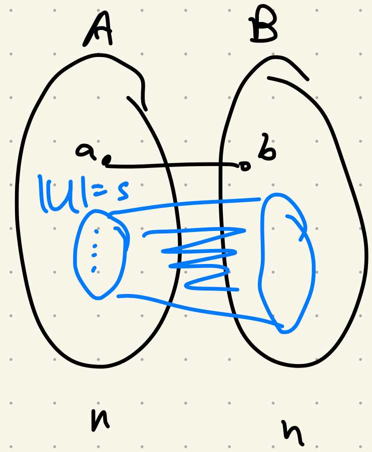
Lecture 24

Recall Kővari-Sós-Turán: $ex(n, K_{s,s}) = O(n^{2-1/s})$

- matching lower bound $\Omega(n^{2-1/s})$ is known only for $t > (s-1)! + 1$. (norm graph)
- Bukh: alternative random alg. constr.
  recently: $t \geq \exp(s)$ ✓.

Let us start w/ showing why $G(n,p)$ is not ideal.

§ Long smooth tail in binomial random constr.



Consider an $n \times n$ -vx b.p. G where each pair $a \in A \sim b \in B$ w/ prob. $p = n^{-1/s}$

$$\mathbb{E}(e(G)) = n^2 p = n^{2-1/s}.$$

- Concentration $\Rightarrow$ w.h.p. $e(G) \geq \frac{1}{2} \mathbb{E}\, e(G)$

We want to $\Pr(K_{s,t} \subseteq G)$ for some t.

For an s-set $U \subseteq A$, write $N^*(U) = \bigcap_{u \in U} N(u)$

$\Rightarrow$ every vx in B falls in $N^*(U)$ w/ prob. precisely $p^s = 1/n$

$\Rightarrow |N^*(U)| \sim Bin(n, 1/n)$, which is distributed roughly as a Poisson r.v. w/ mean 1.

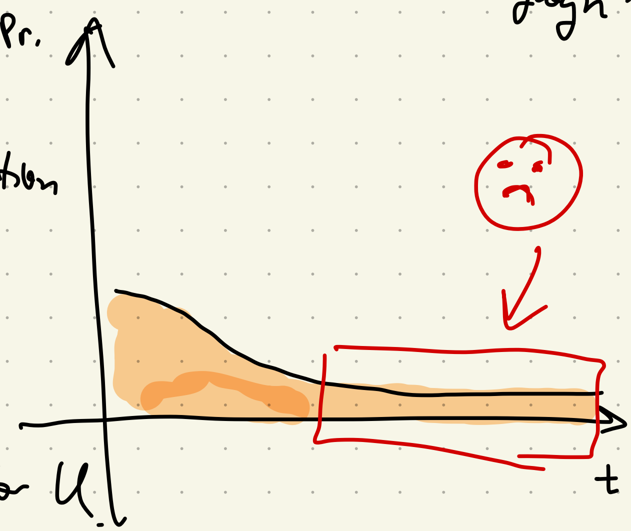
In particular $\Rightarrow$ $\Pr(|N^*(U)| \geq t) \leq 1/t!$ long smooth tail

- By union bound,

$$\Pr(K_{s,t} \subseteq G) \leq 2\binom{n}{s} \cdot \frac{1}{t!} \longrightarrow 0 \quad \text{when } t \geq 10s \frac{\log n}{\log\log n}$$

The order of t cannot be improved: it can be shown that w.h.p. this random graph contains $K_{s,t}$, w/ $t = \frac{s}{10} \frac{\log n}{\log\log n}$.

Conclusion: Even though the r.v. $|N^*(U)|$ has mean 1, its distribution has a long smooth tail and so it is likely that $|N^*(U)|$ is large as there are many choices for U.



§ Random alg. constr.

<u>Idea</u>: Build a graph using a random (bdd deg) polyn. over $\mathbb{F}_q$ and then get rid of (few) copies of $K_{s,t}$ by deletion.

Feature
- locally enjoys the independence as in the binomial random graph and so we can estimate
  - # edges
  - pr. of appearance of small graphs.
- random poly $\Rightarrow$ view $N^*(U)$ as a variety

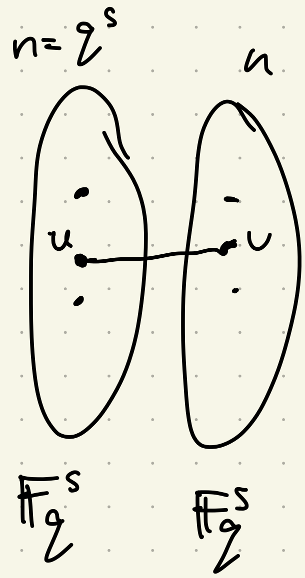
"Magic part": Lang-Weil $\Rightarrow$ $|N^*(U)|$ has a "discontinuity" $\begin{cases} \text{either it is bdd} \\ \text{or } \geq q/2 \end{cases}$

$$\Rightarrow \Pr(|N^*(u)| > C) = \Pr(|N^*(u)| \geq q/2)$$

We have a much better bd. when $q \to \infty$.

Construction Let $s \geq 4$, $d = s^2 - s + 2$

q suff. large prime power, $n = q^s$



Let $f: \mathbb{F}_q^s \times \mathbb{F}_q^s \longrightarrow \mathbb{F}_q$ be a uniform random polyn. on $2s$ variables w./ degree $\leq d$.

Let G be an (n,n)-vx bip. graph w/ each part being a copy of $\mathbb{F}_q^s$ and $u \sim v$ iff $f(u,v) = 0$.

Lem Let f be a unif. $2s$-variate random polyn. w/ deg $\leq d$ over $\mathbb{F}_q$.

$$\Rightarrow \forall u, v \in \mathbb{F}_q^s, \quad \Pr(f(u,v) = 0) = 1/q$$

Pf: · Let $c \in \mathbb{F}_q$ be the constant term in f and set $g = f - c$.

· f unif chosen $\Rightarrow$ c is unif distributed in $\mathbb{F}_q$.

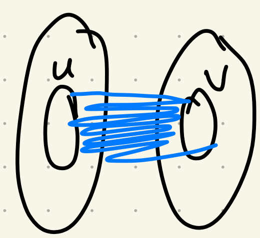
Now, conditioning on the value of $g(u,v)$.

$f(u,v)=0 \iff c = -g(u,v)$, which happens w/ prob. $1/q$.

So, $\mathbb{E}\, e(G) = n^2/q = n^{2-1/s}$

<u>Lem</u> Let f be a unif. $2s$-var. random polyn over $\mathbb{F}_q$.
Let $U, V \subseteq \mathbb{F}_q^s$ be sets of size s and r resp. w/
$s, r \leq \min(\sqrt{q}, d)$

$$\Rightarrow \Pr(\underbrace{f(u,v)=0, \ \forall u\in U, v\in V}_{U \sim V}) = q^{-sr}$$



- Call an s-set U in a partite set of G bad if $|N^*(U)| \geq t$

  We shall control # bad s-sets and do deletion.

- Moments of $|N^*(U)| = \sum_{v\in\mathbb{F}_q^s} \mathbb{1}_{N^*(U)}(v)$

  Linearity of expectation

$$\Rightarrow \mathbb{E}\left(|N^*(U)|^d\right) = \mathbb{E}\left[\sum_{v_1,\dots,v_d\in\mathbb{F}_q^s} \mathbb{1}_{N^*(U)}(v_1)\cdots\mathbb{1}_{N^*(U)}(v_d)\right]$$

$$= \sum_{v_1,\dots,v_d\in\mathbb{F}_q^s} \mathbb{E}\left(\mathbb{1}_{N^*(U)}(v_1)\cdots\mathbb{1}_{N^*(U)}(v_d)\right)$$

$$= \sum_{v_1, \dots, v_d \in \mathbb{F}_q^s} \Pr\left(f(u, v_i) = 0 \ \forall u \in U,\ i \in [d] \right)$$

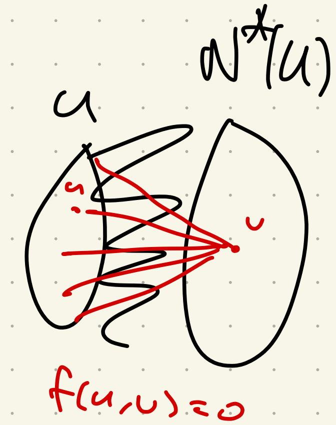
By the previous lem, the summand $= q^{-sr}$ if

$\{v_1, \dots, v_d\}$ has $r \, (\leq d)$ distinct vxs.

Let $M_r = O_d(1)$ be # surjective maps $[d] \to [r]$

$$\Rightarrow \quad \mathbb{E}\left(|N^*(U)|^d\right) = \sum_{r \leq d} \binom{q^s}{r} \cdot M_r \cdot q^{-sr} = O_d(1).$$

- Crucially, by constr. we can view $N^*(U)$ as a variety as follows



$$N^*(U) = \{ v \in \mathbb{F}_q^s : f(u,v) = 0 \text{ for all } u \in U \}$$

$$= \bigcap_{u \in U} \{ f(u, \cdot) = 0 \}$$

Lang-Weil $\Rightarrow$ as a variety

$$|N^*(U)| \begin{cases} \text{either } \lesssim C & (\text{depends only on } s) \\ \text{or } \geq q/2 \end{cases}$$

- Markov ineq $\left(\Pr(X \geq a) \leq \frac{\mathbb{E}X}{a} \right)$

$$\Rightarrow \Pr\Big(\underbrace{|N^*(U)| > C}_{U \text{ bad}} \Big) = \Pr\left(|N^*(U)| \geq q/2\right) = \Pr\left(|N^*(U)|^d \geq \left(\tfrac{q}{2}\right)^d \right)$$

$$\overset{(M)}{\leq} \frac{\mathbb{E}\left(|N^*(U)|^d\right)}{(q/2)^d} = \frac{O_d(1)}{q^d}$$

$\Rightarrow$ expected # bad s-sets $\leq 2\binom{n}{s} \cdot \dfrac{O_d(1)}{q^d}$

$$= O(q^{s-2}) = o(\mathbb{E}\, e(G))$$

Taking $t = C+1 \xRightarrow{\text{deletion}}$ $\overset{2n\text{-vx.}}{K_{s,t}}$-free w/ $\Omega(n^{2-1/s})$ edges

§ Rational Turán exponent.

A rational # $r \in (1,2)$ is a Turán exponent if $\exists$ a bip. graph H s.t. $ex(n,H) = \Theta(n^r)$.

<u>Conj</u> Every rational $r \in (1,2)$ is a Turán exponent.

<u>Ex</u>. $3/2$ ✓ C_4

$2 - 1/s$ ✓ $K_{s,t}$.

<u>Thm</u> (Bukh-Conlon) $\forall$ rational $r \in (1,2)$, $\exists$ a finite fam. graphs $\mathcal{H}$ s.t. $ex(n,\mathcal{H}) = \Theta(n^r)$

The forbidden fam. $\mathcal{H}$ comes from blowups of rooted trees.

<u>Def</u> A rooted tree (T,R) consists of a tree T and an indep. set R as roots. The density ρ_T of

(T,R) is $\dfrac{e(T)}{|T|-|R|}$. For a set of non-roots

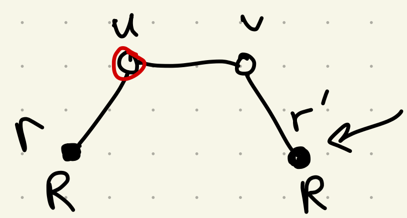
$S \subseteq V(T) \setminus R$, define its density $\rho_S = \dfrac{e(S)}{|S|}$, where

$e(S)$ is # edges in T incident to S.

A rooted tree (T,R) is balanced if $\rho_S \geq \rho_T$ $\forall S \subseteq V(T) \setminus R$.

Rmk: By defn, in a balanced rooted tree, if $|R| \geq 2$ then every leaf in T must be a root. and leaves are evenly distributed

Example



$e(T) = 3$

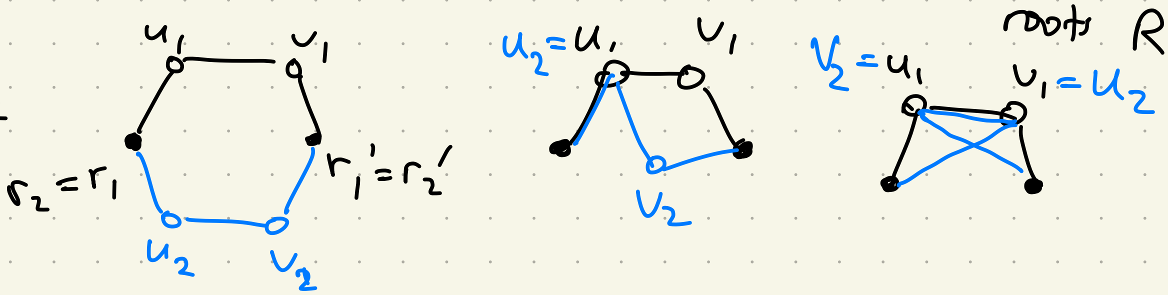
$|R| = 2$

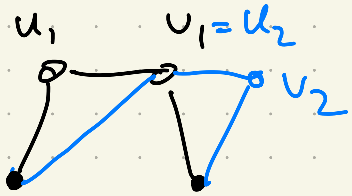
$\rho_T = \dfrac{3}{4-2} = 3/2$

balanced:
- $S = \{u\}$, $\rho_S = \dfrac{2}{1} = 2 \geq \rho_T$
- $S = \{u,v\}$, $\rho_S = 3/2 = \rho_T$.

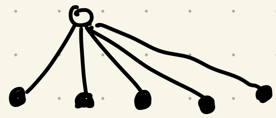
- Given a rooted tree (T,R), the p^{th} power T^p of (T,R) is the fam. all possible unions of p distinct labeled copies of T, all of which agree on the set of roots R.

Ex $p = 2$





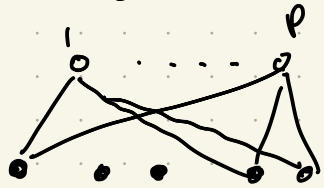
Ex



$|R| = 5$

$\xrightarrow{\;p\;}$

non. degenerate



$K_{|R|, p}$