Lecture 23

Exer. $\forall\, \ell \geq 2$ and non-empty n-vx graph G,

$$\frac{\hom(C_{2\ell}, G)}{\hom(C_{2\ell-2}, G)} \geq \frac{\hom(C_{2\ell-2}, G)}{\hom(C_{2\ell-4}, G)}.$$

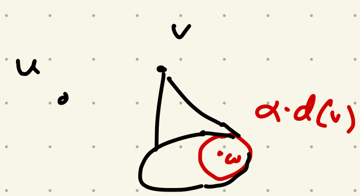
In particular, $\hom(C_{2k-2}, G) \leq n^{\frac{1}{k}} \cdot \hom(C_{2k}, G)^{1-\frac{1}{k}}$.

- $C_0 = K_1$, $C_2 = K_2$.

Hint: consider trace of powers of adj matrix and C-S.

Using this exercise, together w./ some regularisation trick and asymm. Sidorenko's conj for even cycles, from the previous Lem, we get the following version.

Lem Let $k \geq 2$ and $G = (V, E)$ be an n-vx non-empty graph for large n.
Let $\sim$ be a symm. binary relation defined over V s.t.
$\forall u, v \in V$, v has at most $\alpha \cdot d(v)$ many neighbors w s.t.
$u \sim w$. If $\alpha < \frac{1}{n^{1/k} \cdot \log^5 n}$,
$\Rightarrow$ then $\exists$ a C_{2k}-hom $(x_1, \ldots, x_{2k})$ in G
s.t. $\forall\, i \neq j$, $x_i \not\sim x_j$.



Conj (Erdős-Sim) H bip., $\delta(H) = s$, then $\exists\, \varepsilon > 0$ s.t.

$$ex(n, H) = \Omega\left(n^{2 - \frac{1}{s-1} + \varepsilon}\right)$$

Disproved by Janzer using blowup of even cycles.

[J]. $\forall$ even $s \geq 4$, $\forall \delta > 0$, $\exists$ s-reg H $(= C_{2k}[s/2])$

w./ $\quad ex(n, H) = O(n^{2-\frac{2}{s}+\delta})$. $\quad \cdots\cdots (*)$

He also conjectured that the same bound $(*)$ should be true for some s-reg bip. H for every odd $s \geq 3$.

He proved the $s=3$ case. $\exists$ 3-reg bip. H w./

$$ex(n, H) = O(n^{4/3 + o(1)}).$$

<u>Rmk</u> $\quad o(1)$ is needed by considering random construction.

<u>Conj</u> (Grzesik, Janzer - Nagy) $\forall\, 0 \leq \alpha \leq 1$ and H,

if $ex(n, H) = O(n^{2-\alpha}) \Rightarrow$ then $ex(n, H[r]) = O(n^{2-\frac{\alpha}{r}})$

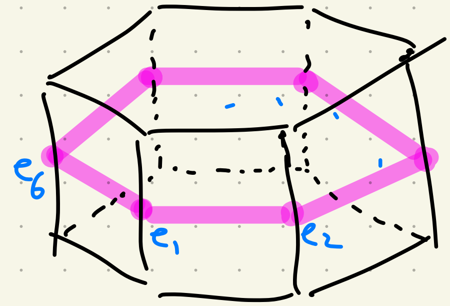
<u>Known</u>:

- True for all trees.
- OPEN for 2-blowup of even cycles

  $ex(n, C_{2k}[2]) = O(n^{3/2 + 1/2k})$?
- true $C_6[2]$. Janzer - Methuku - Nagy.

The 3-reg bip. counterexample of Janzer for $ex(n, H) = \Omega(n^{3/2+\varepsilon})$ has girth 6. Here we give another counterexample w./ girth 4.

• $C_{2k} \square K_2$



$C_6 \square K_2$

Thm $\forall\, k \geq 10$,

$$ex(n, C_{2k} \square K_2) = \Theta(n^{3/2}).$$

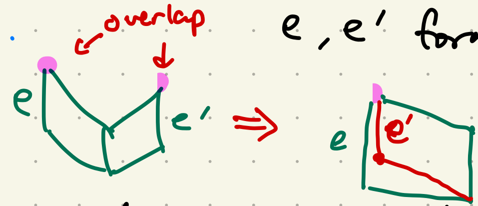
Idea • Let $\Gamma \overset{aux}{=} (V, E)$, $V = E(G)$

E: $e \,\boxed{\sim_P}\, e'$ two vxs form an edge in Γ if e, e' form a C_4 in G.

Γ: C_{2k}: $e_1, \dots, e_{2k}$, $e_i \cap e_j = \phi$ $\forall\, i \neq j$

$\Rightarrow$ $C_{2k} \square K_2$ in G.

Conflict: $e_i \cap e_j \neq \phi$ Ex.

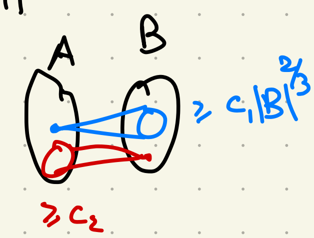


• If there is not much conflict $\Rightarrow$ embedding even cycle without conflict Lem.

• lots of conflict $\Rightarrow$ find an asymm dense subgraph in G and embed $C_{2k} \square K_2$ there using DRC.

Lem Let H be a bip. graph w/ max. deg 3 on one side. Let G be a bip. graph w/ parts A and B and $e(G) = p|A||B|$. Let $c_1, c_2 \geq 2|H|$. Suppose $p \geq c_1 |B|^{-1/3}$ and $p \geq c_2 |A|^{-1}$

$\Rightarrow$ $H \subseteq G$.



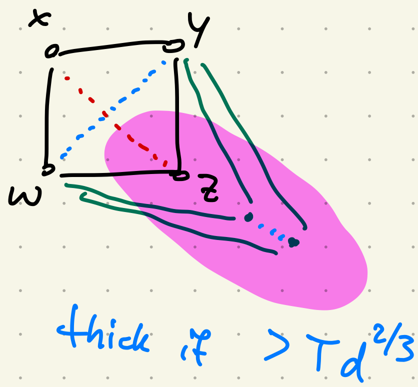
• Now we prove $ex(n, C_{2k} \square K_2) = O(n^{3/2})$.

May assume G · n-vx

· K-almost reg

$K\cdot C n^{1/2} \geq \Delta(G) \geq \delta(G) \geq C n^{1/2} = d$ (C suff. large)

- We call a C_4 $xyzw$ thin if both xz and yw have codeg $\leq T d^{2/3}$, and thick otherwise.



- By supersaturation, G has $\geq c \cdot d^4$ copies of 4-cycles (c depends only on k)

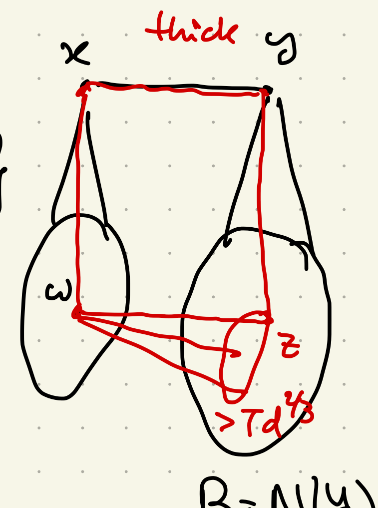
Case 1 $\geq \frac{c}{2} d^4$ C_4's are thick.

- By averaging $\Rightarrow$ $\exists$ an edge xy sitting in $\geq \frac{c d^4/2}{e(G)} \geq \frac{c d^4/2}{K \cdot C n^{3/2}/2}$

wlog, $\geq$ half of these 4-cycles are thick b/c the pair (y,w).

- Let $B = N(y)$

$A = \{ w : \exists$ thick 4-cycle $xyzw$ and $d(w,B) > T d^{2/3} \}$



By defn, for every such thick C_4 w/ $w \in A$, the wz edge lies between A and B

$\Rightarrow$ $e_G(A,B) \geq \#$ thick C_4 on $xy \geq \frac{c d^4}{2 K \cdot C \cdot n^{3/2}}$.

As $|B| = d(y) \leq \Delta(G) \leq K \cdot C \cdot n^{1/2}$

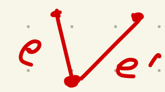
One can check that DRC lem applies $\Rightarrow C_{2k} \triangle K_2 \subseteq G[A,B]$.

Case 2 $\geq cd^4/2$ C_4 are thin.

- Build an aux. graph Γ
  - $V(\Gamma) = E(G)$
  - two adj if they form a thin C_4 in G.

It suffices to embed a $C_{2k} = (e_1, \dots, e_{2k})$ in Γ s.t. $\forall\, i \neq j$

$e_i \cap e_j = \phi$. or $e_i \not\sim e_j$.



Def $\sim$ on $V(\Gamma)$ s.t. $e \sim e'$ if $e \cap e' \neq \phi$ in G

($\heartsuit$)

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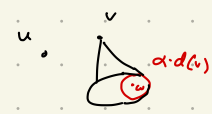


n here is $|\Gamma'| \leq |\Gamma|$
$= e(G)$

- $|V(\Gamma)| = e(G)$
- $e(\Gamma) \geq \#$ thin C_4 in G
  $\geq cd^4/2$

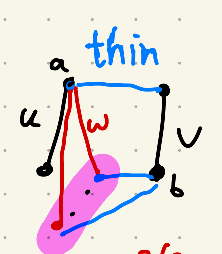
$$\Rightarrow d(\Gamma) \geq \frac{2e(\Gamma)}{|V(\Gamma)|} \geq \frac{cd^4}{e(G)} \geq \frac{2cd^4}{K \cdot C n^{3/2}}$$

- $\Gamma \to \Gamma'$ w/ $\delta(\Gamma') \geq \frac{1}{2} d(\Gamma) \geq \frac{cd^4}{K \cdot C n^{3/2}}$.

Apply ($\heartsuit$) on Γ'. $\forall u, v \in V(\Gamma')$

$\#\{w : w \sim u, w \in N_{\Gamma'}(v)\}$

$\leq d_G(a, b)$



- $w \sim u$
- $w \in N_{\Gamma'}(v)$

As we use only thin C_4 s

$\Rightarrow d_G(a,b) < T d^{2/3}$.

$$\leq \frac{T d^{2/3}}{\delta(\Gamma')}$$

Thus, we need to check $\alpha = \frac{T d^{2/3}}{d_{\Gamma'}(v)} < \frac{1}{e(G)^{1/k} \log^5 e(G)}$

when $k \geq 10$