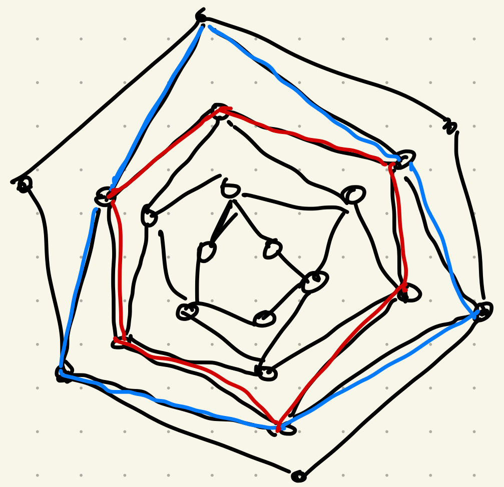
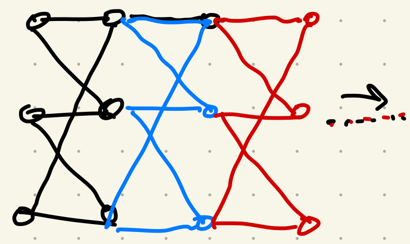
Lecture 22

"Rose": $H_{3,7}$



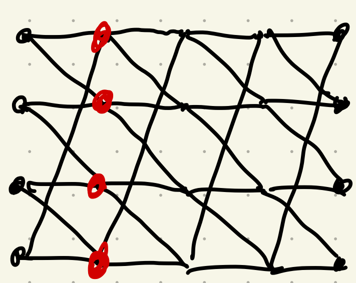
$\cong$

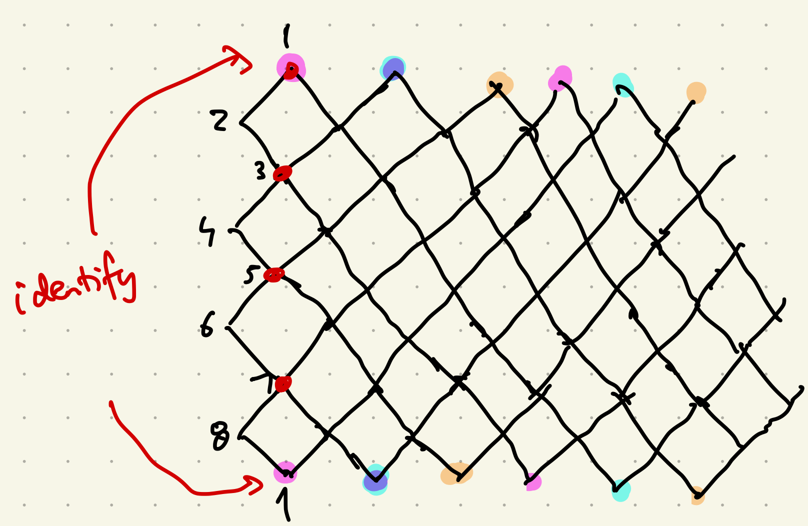


Def: (Cylindrical grid) Let $k, \ell \geq 2$ and define the (k,ℓ)-cylindrical grid $C_{k,\ell}$ as follows.

- $V = \{ x_{i,j} : i \in [k],\ j \in [\ell] \}$
- $E = \{ x_{i,j} x_{i,j+1},\ x_{i,j} x_{i+1,j+1},\ i \in [k],\ j \in [\ell - 1],$ where $x_{k+1,j} = x_{1,j}$ for all $j \in [\ell] \}$

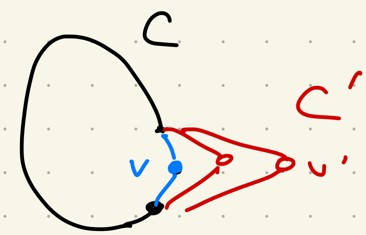
Obs: $C_{t,2t-1}$ contains $G_{t,t}$ as a subgraph.





Thm $ex(n, C_{k,\ell}) = O(n^{3/2})$

Def: Let $R, k \geq 2$. A collection $\mathcal{C}$ of C_{2k} is R-rich if $\forall C \in \mathcal{C}$, $\forall v \in C$, there are $\geq R$ 2k-cycles in $\mathcal{C}$ containing/extending $C - v$.



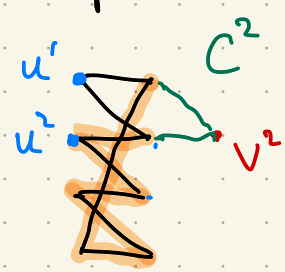
Lem Given $k, R \geq 2$, there exists $C = C(k,R)$ such that $\forall$ n-vx graph G w/ $Cn^{3/2}$ edges contain an R-rich collection of C_{2k}.

Pf (Lem $\Rightarrow$ Thm) Take $R = k \cdot \ell$ and apply Lem

$\Rightarrow$ R-rich collection $\mathcal{C}$.

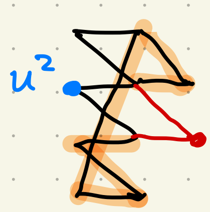
We greedily grow a cylindrical grid.

- Step 1 Take arbitrary $C^1 \in \mathcal{C}$, by R-rich $\exists\, C^2$ extending $C^1 - u^1$.



C^1

Step 2: extend $C^2 - u^2$



<u>Pf</u> (Lem) By supersaturation, G contains $\geq$

$$\alpha \left(d(G)\right)^{2k} \geq \alpha\left(Cn^{1/2}\right)^{2k}, \quad \text{where} \quad \alpha \geq \Omega(k^{-k})$$

- Let $\mathcal{C}_0$ be the coll. of all C_{2k}'s in G.

We keep removing P_{2k-1} that lies in few C_{2k}

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots \supseteq \mathcal{C}_t \text{ (final)} = \mathcal{C}$$

Suppose $\mathcal{C}_i$ is defined. If $\exists\, C \in \mathcal{C}_i$ and a vertex $v \in C$ s.t. # cycles extending $C-v$ is less than R, then remove all cycles in $\mathcal{C}_i$ containing $C-v$.

- Note that each $(2k-1)$-vx path $(C-v)$ is distinct

$$\Rightarrow \quad \#\text{cycles removed} \leq \# P_{2k-1} \cdot R$$
$$\leq R \cdot n \cdot \Delta(G)^{2k-2}$$

- We need $2R \cdot n \cdot \Delta(G)^{2k-2} < \alpha\left(Cn^{1/2}\right)^{2k}$

K-almost reg $\quad 2R \cdot n\left(KCn^{1/2}\right)^{2k-2} < \alpha\left(Cn^{1/2}\right)^{2k}$

$\ldots\ldots\ldots \quad C \geq 2^{k\log k}$

§ Even cycle embedding without conflict via dyadic partitioning.

Janzer 1) "Rainbow Turán # of even cycles, ..."

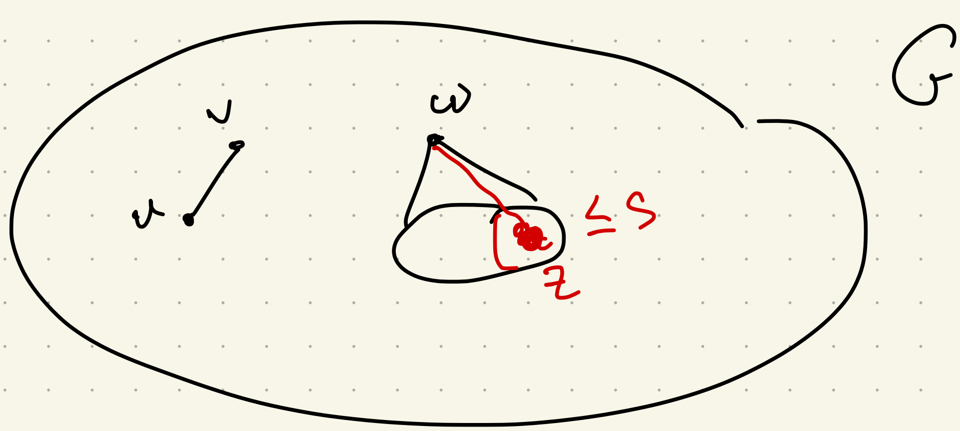
2) "Dispf of a conj. Erdős - Simonovits ....."

Lem 1 (Lem 2.5 in 1)) Let $k \geq 2$ and G be a graph w./ a symm. binary relation $\sim$ defined over V^2.

Suppose $\forall (u,v) \in V^2$ and $\forall \omega \in V$, ω has at most s neighbors z s.t. $(u,v) \sim (\omega, z)$

$\Rightarrow$ Then # C_{2k}-homomorphisms $(x_1, \dots, x_{2k})$ s.t. $(x_i, x_{i+1}) \sim (x_j, x_{j+1})$ for some $i \neq j$ is

$$\leq 32k \sqrt{k s \cdot \Delta(G) \cdot \hom(C_{2k-2}, G) \hom(C_{2k}, G)}.$$



Advantage of dyatic partition/pigeonholing

1): For each part, we can treat all members there as if they are the same

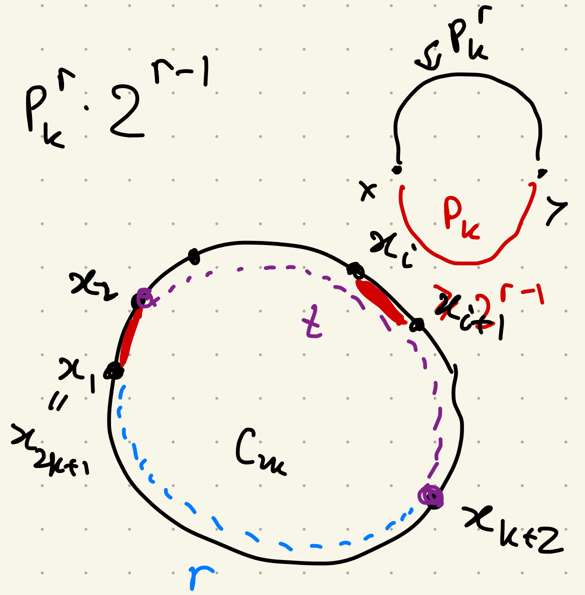
2): # parts is logarithmic, very small.

<u>Pf</u>: For $r \in \mathbb{N}$, let

- p_k^r = # P_k-homomorphisms whose endpts x,y satisfy $\hom_{x,y}(P_k, G) \in [2^{r-1}, 2^r)$
  (length-(k−1) walk)
- p_{k+1}^r = # P_{k+1}-hom. whose endpts x,y satisfy $\hom_{x,y}(P_{k+1}, G) \in [2^{r-1}, 2^r)$
  (length-k walk)

By defn.
$$\hom(C_{2k-2}, G) \geq \sum_r p_k^r \cdot 2^{r-1}$$

$$\hom(C_{2k}, G) \geq \sum_r p_{k+1}^r \cdot 2^{r-1}.$$



- For $r, t \in \mathbb{N}$, let

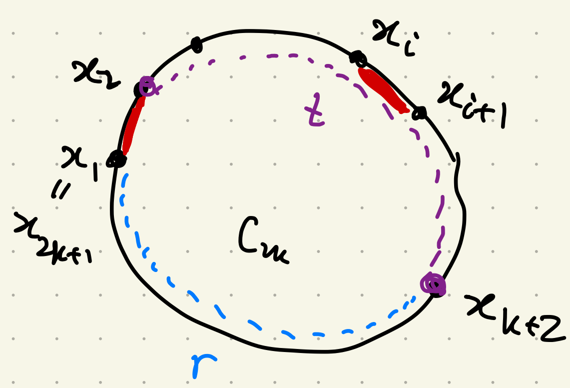
$$\gamma_{r,t} = \# \ C_{2k}\text{-hom. } (x_1, x_2, \ldots, x_{2k})$$

s.t.
- $\hom_{x_1, x_{k+2}}(P_k, G) \in [2^{r-1}, 2^r)$
- $\hom_{x_2, x_{k+2}}(P_{k+1}, G) \in [2^{t-1}, 2^t)$
- $\exists\ 2 \leq i \leq k+1,\ x_1x_2 \sim x_ix_{i+1}$

Conflict occurs at $x_1x_2 \sim x_ix_{i+1}$

<u>Total # BAD</u> C_{2k}-hom $\leq \sum_{r,t \geq 1} \gamma_{r,t}$

Let us bound $\gamma_{r,t}$ from above in two diff. ways.



① $\gamma_{r,t} \leq P_k^r \cdot \Delta(G) \cdot 2^t$

- first choose P_k-hom of type r $\rightsquigarrow$ (so x_1, x_{k+2} fixed) $\leq P_k^r$ choices
- Choose x_2 $\longrightarrow$ $\leq \Delta(G)$
- Choose P_{k+1}-hom with given ends x_2, x_{k+2} (blue then purple) $\rightsquigarrow$ $\leq 2^t$

② $\gamma_{r,t} \leq P_{k+1}^t \cdot k \cdot s \cdot 2^r$.

(purple then blue)

- first P_{k+1}-hom of type t $\rightsquigarrow$ x_2, x_{k+2} fixed $\leq P_{k+1}^t$.
- Choose x_1, $\leq k \cdot s$
- Choose P_k-hom w/ given ends x_1, x_{k+2}

WANT Bound

$$\sum_{r,t \geq 1} \gamma_{r,t} \overset{①}{\leq} \sum_{r,t} P_k^r \cdot \Delta(G) \cdot 2^t$$

$$= \Delta(G) \sum_{r,t} P_k^r \cdot 2^r \cdot 2^{t-r}$$

C_{2k-2} up to a factor 2

Suppose $t-r \leq q$ to be determined.

$$\sum_{r,t:\ t-r\le q} \gamma_{r,t} \lesssim \Delta(G)\cdot \sum_{r,t:\ t-r\le q} p_k^r \cdot 2^r \cdot 2^q$$

$$= \Delta(G)\cdot 2^q \sum_{r,t:\ t-r\le q} p_k^r\, 2^r$$

$$\approx \Delta(G)\, 2^q \cdot \hom(C_{2k}, G) = \mathrm{I}$$

$$\sum_{r,t:\ t-r>q} \gamma_{r,t} \overset{(2)}{\le} \sum_{r,t:\ t-r>q} p_{k+1}^t \cdot k\cdot s\cdot 2^r \qquad 2^r = 2^t \cdot 2^{-(t-r)}$$

$$\lesssim k\cdot s\cdot 2^{-q} \sum_{r,t:\ t-r>q} p_{k+1}^t \cdot 2^t$$

$$\le k\cdot s\, 2^{-q} \hom(C_{2k}, G) = \mathrm{II}$$

Total bad $\le \sum_{r,t} \gamma_{r,t} \le \mathrm{I} + \mathrm{II}$

Worst case $\mathrm{I} = \mathrm{II} \Rightarrow 2^q \approx \sqrt{\dfrac{k s \hom(C_{2k},G)}{\Delta(G) \hom(C_{2k-2},G)}}$

$\cdots \longrightarrow$  .