Lecture 21

- Another application of [iterative C-S + Sidorenko]

Turán # of the hypercube = Janzer - Sudakov.

<u>Thm</u> $\forall$ integer $d \geq 3$,

$$ex(n, Q_d) = O_d\left(n^{2 - \frac{1}{d-1} + \frac{1}{(d-1)2^{d-1}}}\right)$$

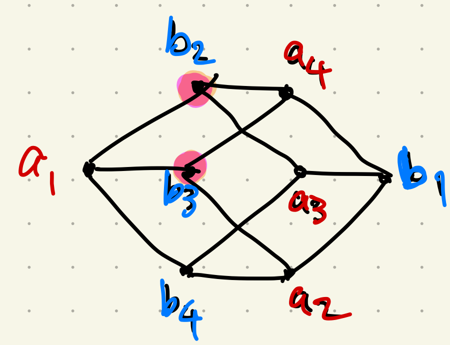
<u>Rmk</u>: Note that Q_d is $K_{3,3}$-free and d-reg,

(hence Füredi: $O(n^{2-1/d})$). Exponent here beats $2 - 1/d$.

We shall illustrate the method via $d=3$ case, i.e.

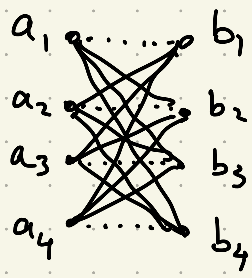
$$ex(n, Q_3) = O\left(n^{2-3/8}\right)$$

- Up to symm., $\forall\, a_i, a_j$, $i \neq j$



the Q_3-hom mapping a_i, a_j to the same vertex is the 'largest' degenerate one; let D_2 be the count.

Let D_i, $i \in [4]$, be the hom. count of Q_3-hom mapping i vxs of $\{a_1, \ldots, a_4\}$ (or $\{b_1, \ldots, b_4\}$) to one same vertex.



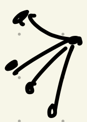
$D_1 = \hom(Q_3, G)$, $D_4 = \hom(S_4, G)$

Idea : If there is injective (non-degenerate) Q_3-hom, ☺.

As D_2 is the count of the largest deg. one, we may assume $D_2 \approx Q_3 = D_1$

$$\downarrow \text{C-S}$$

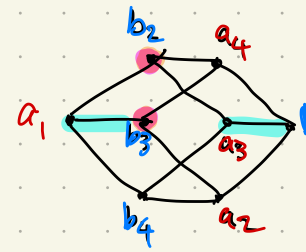
$$D_3 \approx Q_3 = D_1$$



$$\downarrow \text{C-S}$$

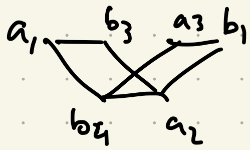
$$n^5 p^4 \geq S_4 = D_4 \approx Q_3 \overset{\text{Sidorenko}}{\geq} n^8 p^{12}$$

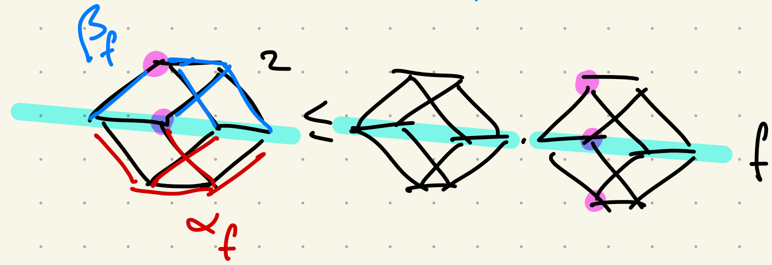
$$\Rightarrow \quad p \leq n^{-3/8} \quad \lightning.$$



pf : Claim: $D_2^2 \leq D_1 \cdot D_3$

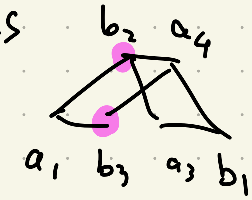
- Let f be a hom. of $a_1 b_3$ and $a_3 b_1$
- Let α_f be # extensions of f to a hom. of





$$\left(\sum_f \alpha_f \beta_f\right)^2 \leq \left(\sum_f \alpha_f^2\right)\left(\sum_f \beta_f^2\right)$$

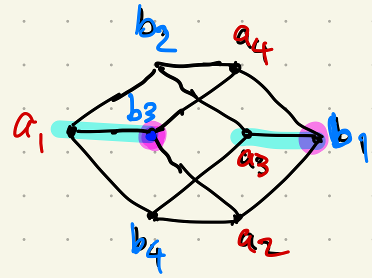
- Let β_f be # extensions of f to a hom. of



such that b_2, b_3 got mapped to the same vertex.

Note that by symm. $\sum \alpha_f^2 = D_1$, $\sum \beta_f^2 = D_3$

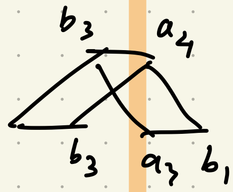
and $\sum \alpha_f \beta_f = D_2$.



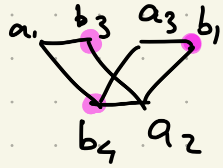
Claim $D_3^2 \le D_2 \cdot D_4$

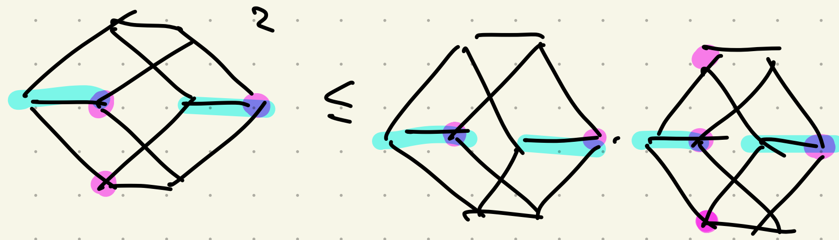
Pf: • Let f be a hom. of $a_1 b_3 \cup a_3 b_1$ such that b_3, b_1 got mapped to the same vertex

• Let α_f be # extensions of f to a hom. of



• Let β_f be # extensions of f to a hom. of





$D_3 = \sum_f \alpha_f \beta_f$

$D_2 = \sum_f \alpha_f^2$

$D_4 = \sum_f \beta_f^2$

such that b_4, b_3 got mapped to the same vertex

Putting them together, we have

$$D_2^2 \le D_1 \cdot D_3 \quad \cdots (1)$$

$$D_3^2 \le D_2 \cdot D_4 \quad \cdots (2)$$

$D_1 = Q_3$

$Q_4 = S_4$

We assume $D_2 \gtrsim Q_3 = D_1$ ← a positive fraction

K-almost $\Delta(G) \le K \cdot n \cdot p$

$(1) \Rightarrow D_3 \geq \dfrac{D_2^2}{D_1} \gtrsim D_1$

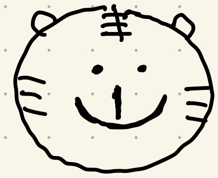
$(2) \Rightarrow D_4 \geq \dfrac{D_3^2}{D_2} \gtrsim D_1 = Q_3 \geq n^8 p^{12}$ (Sidorenko)

$K^4 n^5 p^4 \geq n \cdot \Delta(G)^4 \geq S_4 = D_4$

$\cdots \Rightarrow \quad p < \text{const.}\, n^{-3/8} \quad \lightning$

This means $\quad D_2 \leq \frac{1}{24} D_1 = \frac{1}{24} \hom(Q_3, G)$

- By symm, total # deg. Q_3-hom is $\leq 12 \cdot D_2 \leq \frac{1}{2} \hom(Q_3, G)$.



- Kim - Lee - L. - Tran $\begin{cases} \text{- rainbow cycle} \\ \text{- supersaturation} \end{cases}$

- Janzer - Sudakov. can also do $\begin{cases} \text{- hypercube} \\ \text{- reflexitive graphs} \end{cases}$

[KLLT]: $\forall$ n-vertex G w/ ave. deg $d \geq 2 \cdot 10^5 k^3 n^{1/k}$ contains $\geq \frac{1}{2} (2^{12} k)^{-k} \cdot d^{2k}$ copies of C_{2k}.

[J-S] $\forall\ 1 \leq \ell < k/2$, let $H_{\ell,k}$ be the bip. graph w/ parts $\binom{[k]}{\ell}$ and $\binom{[k]}{k-\ell}$ and two vertices $S \in \binom{[k]}{\ell}$, $T \in \binom{[k]}{k-\ell}$ form an edge iff $S \subseteq T$

Let $d = \binom{k}{\ell}$, $\exists\ \varepsilon = \varepsilon(\ell, k) > 0$ s.t.

$$ex(n, H_{\ell,k}) = O\left(n^{2 - 1/d - \varepsilon}\right)$$

Note: $H_{1,4} = Q_3$.

Def: $ex^*(n, \mathcal{H})$ = max # edges in a properly edge-colored n-vx graph G w/ no rainbow copies of $H \in \mathcal{H}$.

$\mathcal{C}$ = fam. of all cycles.

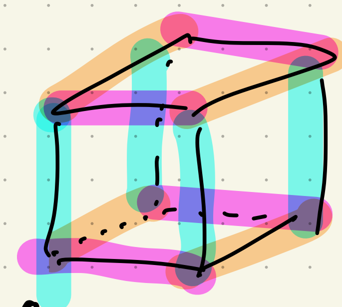
- $ex(n, \mathcal{C}) = n-1$

- Das - Lee - Sudakov: $ex^*(n, \mathcal{C}) \geq n \log_2 n$

  [KLLT, JS]: $ex^*(n, \mathcal{C}) = O(n \log^2 n)$

OPEN: Is $ex^*(n, \mathcal{C}) = \Theta(n \log n)$?

Lower bound: Consider $G = Q_t$ hypercube



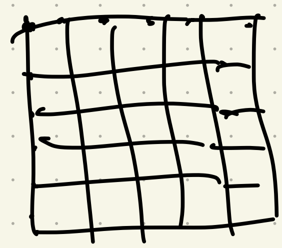
$2^t = n$, $t = \log_2 n$, $e(G) = n \log_2 n$

Suffices to find a proper edge coloring of Q_t w/ no rainbow cycle.

Coloring by direction of the edge. and note that every cycle in Q_t contains ≥ 2 edges of the same direction.

§ An application of supersaturation

Let $G_{t,t}$ be the $t \times t$ grid.



$G_{t,t}$ $t=6$

Clearly as $C_4 \subseteq G_{t,t} \Rightarrow$

$$ex(n, G_{t,t}) \geq \Omega(n^{3/2})$$

<u>Thm</u> (Bradač - Janzer - Sudakov - Tomon)

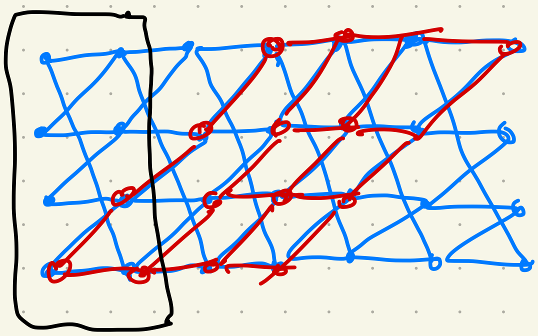
$\forall t \in \mathbb{N}$, $ex(n, G_{t,t}) = \Theta_t(n^{3/2})$

<u>Rmk</u>: The constants is $2^{O(t^5)}$.

Here we give a much simpler pf, showing an upper bound of $O(2^{t \log t} \cdot n^{3/2})$.

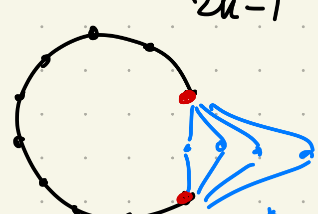
<u>Idea</u>:

$k=4$

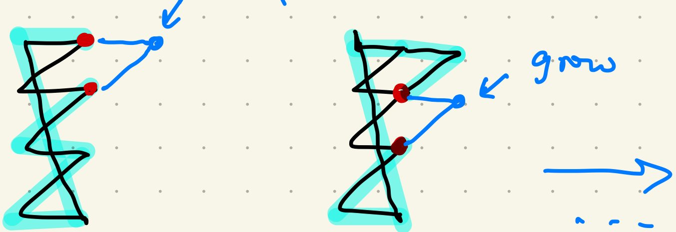


C_{2k} $\longrightarrow$ grow.

$\forall P_{2k-1}$



1) Find a good collection $\mathcal{C}$ of C_{2k}:



$C' \in \mathcal{C}$