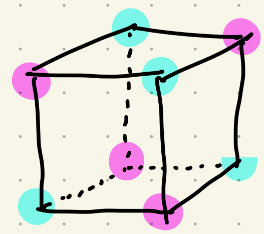
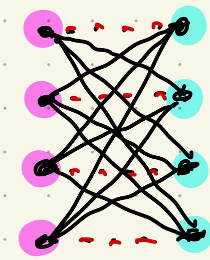
Lecture 20

§ 3-dim cube Q_3



- 3-reg, bip
- Füredi $\Rightarrow$ $ex(n, Q_3) = O(n^{2-1/3}) = O(n^{5/3})$
- Note that $Q_3 = K_{4,4} \setminus PM$



$K_{3,3}$-free !

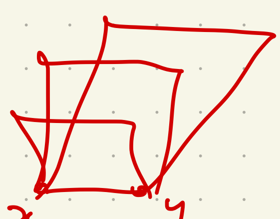
$= n^{2-2/5}$

Thm $ex(n, Q_3) = O(n^{8/5})$

Rmk: For lower bd, we only know $\geq n^{3/2}$

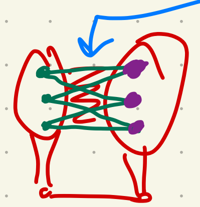
Idea:

- Supersaturation $\Rightarrow C_4$
- $\exists$ edge xy



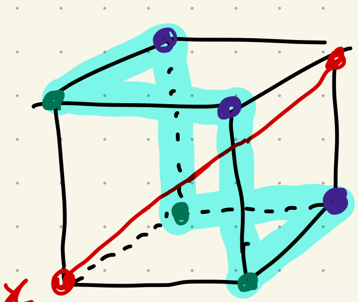
sitting in many C_4.

$\Longleftrightarrow$



- Find C_6 in $G[N(x), N(y)]$

$\Rightarrow Q_3^+$



Pf: By E-Sim. regularisation trick, may assume G is K-almost reg. i.e.

Recall n-vx G
$d(G) \geq 2\sqrt{n}$
$\Rightarrow \geq \frac{d(G)^8}{8}$
copies of C_4

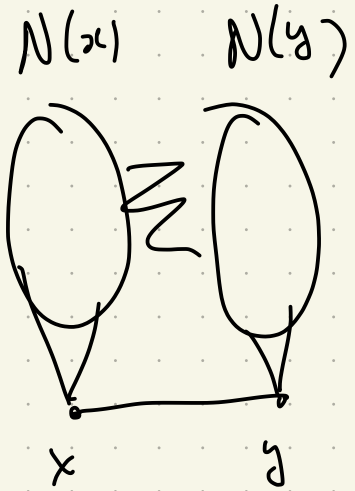
$$Cn^{3/5} \leq \delta(G) \leq \Delta(G) \leq K\cdot\delta(G) \leq K\cdot Cn^{3/5}.$$

$(C \gg K)$

- Sup-sat $\Rightarrow$ $\# C_4 \geq d^4/8$

$d = d(G)$

$\Rightarrow \exists\, xy$ edge in $\geq \frac{\# C_4}{e(G)} \geq \frac{d^4/8}{dn/2}$ many C_4.s $\geq \frac{C^3}{4} n^{4/5}$



$(*) \ldots > ex(2Kd, C_6)$

$G' = G[N(x), N(y)]$ (G bip. $N(x) \cap N(y) = \phi$)

$\# C_4 \supseteq \{x,y\} = e(G')$

G' has $\leq 2\Delta(G) \leq 2Kd$ vxs

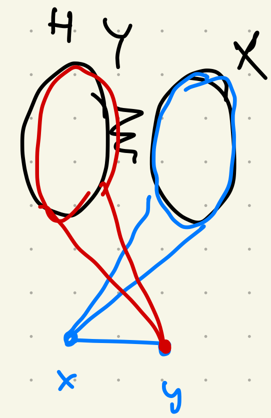
So $(*) \Rightarrow G'$ has $C_6 \Rightarrow G$ has Q_3^+ $\square$

- Erdős - Simonovits reduction trick.

Given bip. H on part. set $X \cup Y$,

$K_2 * H$: new bip.

- add edge xy
- add all edges x, X
- —— " —— y, Y.



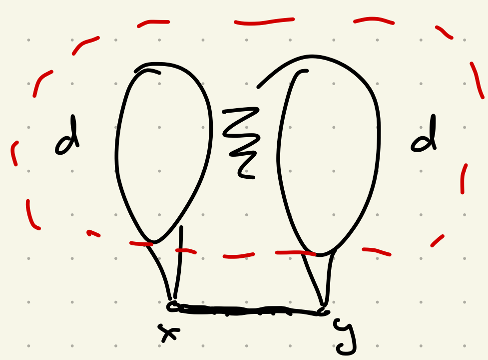
$\boxed{\text{e.g.} \quad Q_3^+ = K_2 * C_6}$

Suppose we want upp bd $ex(n, K_2 * H)$.

and we know $ex(n, H) = O(n^\alpha)$

where $1 < \alpha < 2$.

Try d-reg



find H in G'

$G' = G[N(x), N(y)]$

$e(G') = \#\{C_4 \supseteq \{x, y\}\}$

$$\gtrsim \frac{d^4}{nd/2} \approx \frac{d^3}{n} \gg d^\alpha \gtrsim ex(2d, H)$$

$$d^{3-\alpha} \gg n \qquad d > n^{\frac{1}{3-\alpha}}$$

$$\Rightarrow \text{if } e(G) > dn \approx n^{1 + \frac{1}{3-\alpha}} = n^{\frac{4-\alpha}{3-\alpha}}$$

$$\Rightarrow K_2 * H$$

Thm (Erdős–Sim)

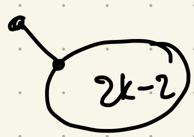
$$\mathrm{ex}(n,H) \le O(n^{\alpha}) \implies \mathrm{ex}(n, K_2 * H) = O\left(n^{\frac{4-\alpha}{3-\alpha}}\right)$$

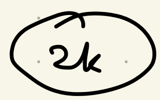
§ 3rd pf of even cycles via
• Sidorenko
&
• iterative Cauchy-Schwarz.

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$$

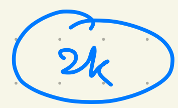
Idea : • If most of C_{2k}-hom are non-deg. ☺

• So may assume positive fract. of C_{2k}-hom are degenerate:

If 

$\approx$ 

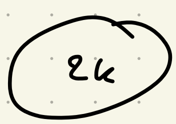
then

using • iterative C-S $\implies$  $\approx$ 

S_k

in fact the most degenerate C_{2k}-hom, i.e. S_k-hom.
is comparable to C_{2k}-hom

Sidorenko

$$n^{k+1}p^k \approx n \cdot \Delta(G)^k \ge$$  $\approx$ 

$\ge n^{2k}p^{2k}$

S_k

$$\implies p < n^{\frac{1}{k}-1}$$

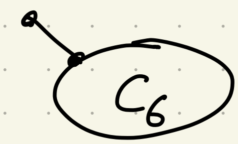
Illustrate this idea w/ $k=4$.

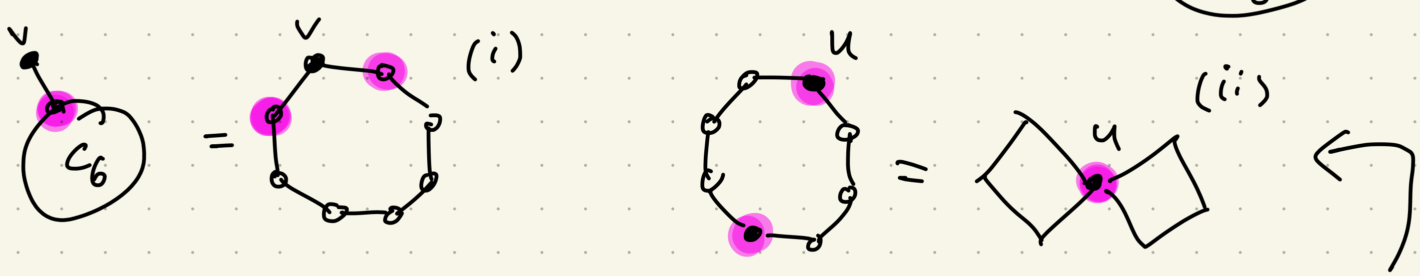
<u>Show</u> $ex(n, C_8) = O(n^{1+1/4})$

<u>NTS</u> Given n-vx G · K-almost reg $\Rightarrow C_8$

· $d \geq C \cdot n^{1/4}$

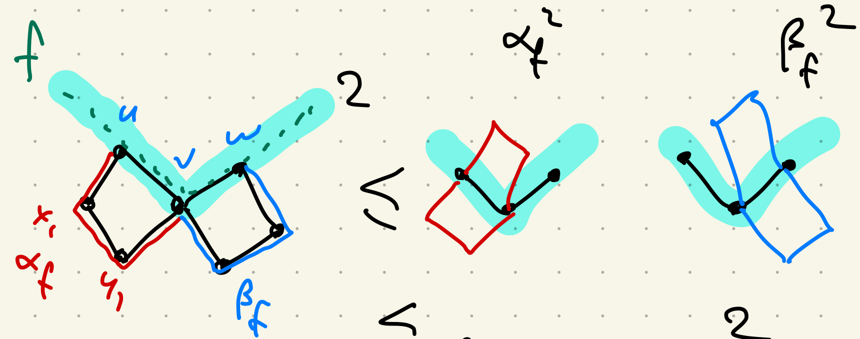
· $p = \frac{d}{n} \approx Cn^{-3/4}$

1) We can bd all deg. ones by 

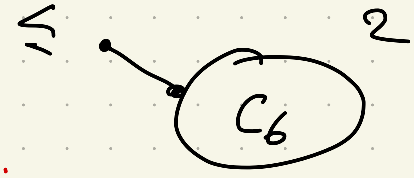


· There are only two types of largest deg. C_8-hom

Shall bd (ii) by (i)

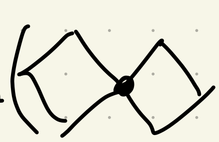


f: P_3-hom in G
uvw



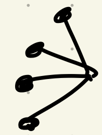
α_f : # extensions of f
$u\,x_1 y_1 v$ P_4-hom w/ endpts u, v.

β_f : # ext. of f P_4-hom w/ endpts v, w

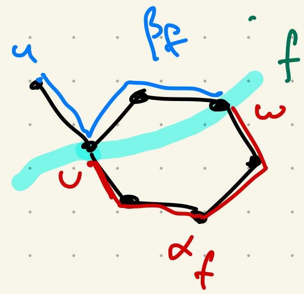
hom( , $G)^2 = \left(\sum_f \alpha_f \cdot \beta_f\right)^2 \leq \left(\sum_f \alpha_f^2\right) \cdot \left(\sum \beta_f^2\right)$

- Thus, we may assume

$$\hom\left(C_6 \right) = \Omega\left(\hom(C_8) \right)$$

- Now we show $C_6 \approx C_4 \approx$ 

$$C_6^{\,2} \leq C_8 \cdot C_4$$



$$\| $$

$$\left(\sum_f \alpha_f \, \beta_f \right)^2 \leq \left(\sum \alpha_f^2 \right) \cdot \left(\sum \beta_f^2 \right)$$

f : v w

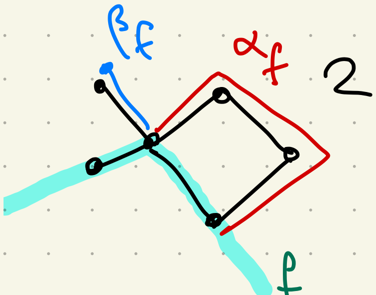
$$= C_8 \cdot \boxed{C_4}$$

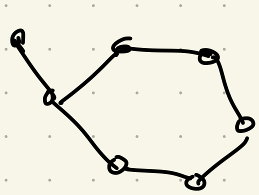
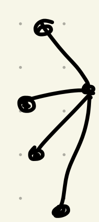
α_f : # ext. P_5-hom w/ endpt v, w

β_f : # ext P_4-hom as in picture.

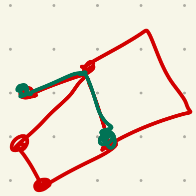
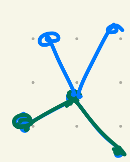
$\Rightarrow$ as $C_6 = \Omega(C_8)$

we get that $C_4 = \Omega\left(C_6 \right) = \Omega\left(C_8 \right)$

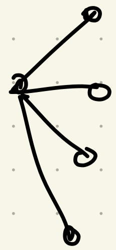
- 

$\leq$  $\cdot$ 

$$\left(\sum_f \alpha_f \cdot \beta_f \right)^2 \leq \left(\sum \alpha_f^2 \right) \cdot \left(\sum \beta_f^2 \right)$$

$=$  $\cdot$  $\leq$

Chase back $\dashrightarrow$

$n^5 p^4 \geq$  $= \Omega(C_8)$

Sidorenko

$\geq n^8 p^8$

$p < n^{-3/4}$