Lecture 19

- A related conj of Kohayakawa - Nagle - Rödl - Schacht : for all H in locally dense host graphs. See [Lee] On some graph densities in locally dense graphs.

§ Equiv. E-S conj and Sidorenko's conj via tensor power trick.

<u>Conj</u> (Erdős - Simonovits) $\forall$ bip. H, $\exists\ c>0$, $\alpha>0$ and n_0 s.t. TFH for all n-vx graphs G w./ $n \geq n_0$.
If G has $\geq n^{2-c}$ edges $\Rightarrow$ then it contains $\geq \alpha n^{|H|} \cdot p^{e(H)}$ copies of H, where $p = \frac{2e(G)}{n^2}$.

We shall prove that E-S conj is equiv. to S. conj

The easy direction

[Sidorenko $\Rightarrow$ E-S] : Fix a bip. H, by Sidorenko's conj.

$\hom(H,G) \geq n^{|H|} p^{e(H)}$.

- $\#$ non-injective H-hom. $\leq n^{|H|-1}$ if $< \frac{1}{2} n^{|H|} p^{e(H)}$

$\Longleftrightarrow$ $p^{e(H)} > \frac{2}{n}$

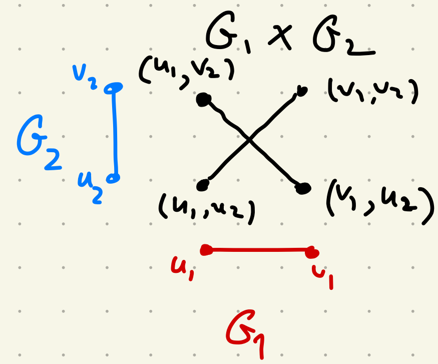
$p > \left(\frac{2}{n}\right)^{-\frac{1}{e(H)}}$

So we can take

$$c = -\frac{1}{e(H)} \quad \text{and} \quad \alpha = \frac{1}{2|\mathrm{Aut}(H)|}.$$

Def: Given graphs G_1 and G_2, the **tensor product** of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph w/ vertex set $V(G_1) \times V(G_2)$ and two vxs (u_1, u_2), (v_1, v_2) are adj. iff $u_1 \underset{G_1}{\sim} v_1$ and $u_2 \underset{G_2}{\sim} v_2$

We write $G^{\times k}$ for the tensor power $\underbrace{G \times G \times \cdots \times G}_{k \text{ times}}$.

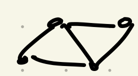


The direction that E-S conj $\Rightarrow$ Sidorenko's conj.

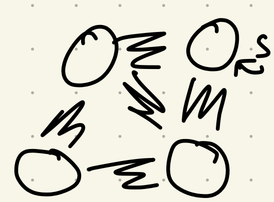
Idea: if Sidorenko fails for some H and G

$\Rightarrow$ E-S fails for H and some G'

use tensor power to amplify



Recall the s-blowup of G, denoted $G[s]$,



Ex. $\forall H, G, s \in \mathbb{N}$, $t(H, G) = t(H, G[s])$.

Prop $\forall H, G, k \in \mathbb{N}$, $t(H, G^{\times k}) = t(H, G)^k$

Pf: It suffices to prove

$$\mathrm{Hom}(H, G \times G) \cong \mathrm{Hom}(H, G) \times \mathrm{Hom}(H, G). \quad (*)$$

Indeed, then $t(H, G\times G) = \dfrac{\hom(H,G)^2}{|G\times G|^{|H|}} = t(H,G)^2$.

- For $(*)$ consider $f \longmapsto (\pi_1\circ f, \pi_2 \circ f)$ where π_i is the projection map to i^{th} coordinate.

$\forall\, x \sim_H y$, $\quad f(x) \underset{G\times G}{\sim} f(y)$

by def of $G\times G \Rightarrow \quad \pi_i\circ f(x) \sim_G \pi_i \circ f(y)$. ☺

Pf (E-S $\Rightarrow$ S.)

Suppose Sidorenko's conj fails for H and G, i.e.

$\exists \varepsilon > 0$ s.t. $\boxed{t(H,G) < (1-\varepsilon)\, t(K_2, G)^{e(H)}}$

- Prop $\Rightarrow$ $t(H, G^{\times k}) = t(H,G)^k < (1-\varepsilon)^k\, t(K_2, G^{\times k})^{e(H)}$ $(\heartsuit)$

- By taking large enough blowup $G' = G^{\times k}[S]$, we may assume G' has edge density $p := t(K_2, G') \geq \frac{1}{2}$.
- By E-S conj, we have # copies of H, hence also $\hom(H, G')$, is $\geq c\, |G'|^{|H|}\cdot p^{e(H)}$

Thus $\quad t(H, G') \geq c\cdot p^{e(H)}$.

$(\heartsuit) \Rightarrow \quad t(H,G') \overset{\text{ex.}}{=} t(H, G^{\times k}) < (1-\varepsilon)^k\, \underbrace{t(K_2, G')}^{e(H)} = (1-\varepsilon)^k p^{e(H)}$

Taking k suff. large s.t. $(1-\varepsilon)^k < c$, we get $\lightning$.

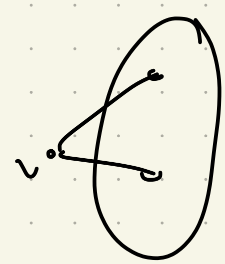
<u>Rephrase E-S conj</u>.

[E-S conj$'$] $\forall H$ bip, $\exists\, \alpha > 0$ s.t. $\forall\, p \in (0,1)$ and $\forall$ suff large graph G on n vxs and $\geq \frac{1}{2} n^2 p$ edges $\Rightarrow$ contains $\geq \alpha n^{|H|} p^{e(H)}$ copies of H.

Let us prove that C_4 satisfy Sidorenko's conj via E-S conj. That is, it suffices to prove the following supersaturation result.

<u>Prop</u> $\forall$ n-vx G w/ large n and $d(G) \geq 2\sqrt{n}$

$\Rightarrow \geq \frac{d(G)^4}{8}$ copies of C_4 in G.



<u>Pf</u>: Let $d = d(G)$

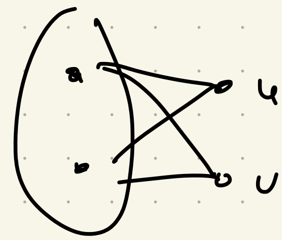
- $\#$ cherries $K_{1,2}$ in $G \geq \sum_{v \in V(G)} \binom{d(v)}{2}$

$$\overset{\text{Jensen's ineq}}{\geq} n \binom{\frac{1}{n}\sum d(v)}{2} = n \binom{d}{2}.$$

- average codeg. in G is

$$D = \frac{1}{\binom{n}{2}} \sum_{u,v \in \binom{V(G)}{2}} d(u,v) = \frac{\# K_{1,2}}{\binom{n}{2}} \geq \frac{d(d-1)}{n-1} \geq 2$$

- $\#C_4 \geq \sum_{uv \in \binom{V(G)}{2}} \binom{d(u,v)}{2} \overset{\text{Jensen}}{\geq} \binom{n}{2} \cdot \binom{D}{2} \geq \frac{d^4}{8}$



Ex. Prove Sidorenko's conj for $K_{s,t}$ via E-S conj.

$\forall\, t \geq s \geq 2$, $\forall$ n-vx G w/ $\frac{1}{2}n^2 p$ edges

$$\#\text{ labeled copies of } K_{s,t} \text{ in } G \geq \frac{1}{s!\,t!\,2^{ts+1}} \cdot n^{t+s} p^{ts}.$$

Ex Prove Sidorenko's conj for paths by showing that

$$\hom(P_{k+1}, G) \geq 2^{-(2k+1)} \cdot n^{k+1} p^{k}, \quad \text{where } p = t(K_2, G) = \frac{2e(G)}{n^2}$$

Hint: Induction on n

- if $\exists$ vxs of deg $< \frac{1}{4}np$, remove
  $\cdots \Rightarrow G'$, apply IH on G'.
- o.w. high min deg, greedily find P_{k+1} Hom.

§ Turán number of even cycles via homomorphism inequalities.

Baby case $\quad ex(n, C_6) = O(n^{4/3})$

$ex(n, C_{2k}) = O(n^{1+1/k})$

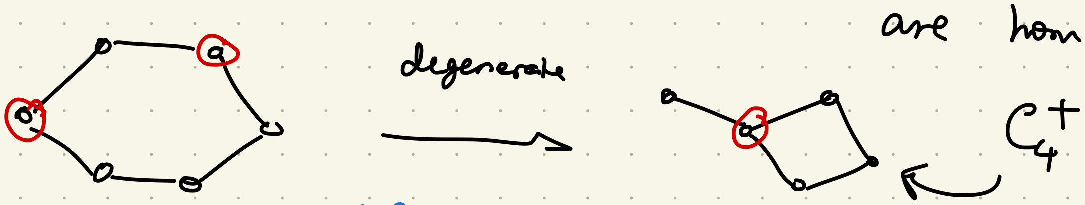
The idea is to try to prove that many C_6-hom. in graphs w./ $\gg n^{4/3}$ edges are injective. $\Rightarrow$ a copy of C_6.

Pf: Let G be an n-vx graph w./ $e(G) \geqslant C n^{4/3}$ for suff. large $C > 0$.

Write $p = \dfrac{2e(G)}{n^2} = \dfrac{2C}{n^{-2/3}}$

• Sidorenko's conj $\Rightarrow$ $\hom(C_6, G) \geqslant n^6 p^6$

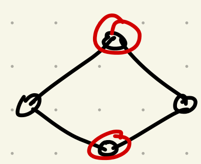
Suppose all C_6-hom are degenerate. $\Rightarrow$ all of them are hom. of



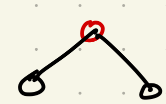
$\Delta(G) \leq Kd = Knp$

Assuming G is K-almost reg, we have

$$\hom(C_4, G) \geqslant \frac{\hom(C_6, G)}{\Delta(G)} \geqslant \frac{1}{K} n^5 p^5$$



Among these, $\leq \hom(K_{1,2}, G)$



many of them $\leq n\,\Delta(G)^2 \leq K^2 n^3 p^2 < \frac{1}{2}\hom(C_4, G)$

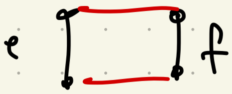
are degenerate CHECK

$K^2 n^3 p^2 < \frac{1}{2} K n^5 p^5$

So we have $\geqslant \frac{1}{2K} n^5 p^5$ many copies of C_4.

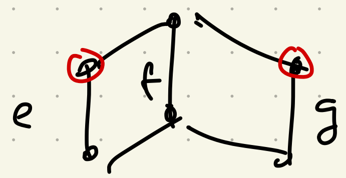
Build auxiliary Γ on vertex set $E(G)$

and $e \sim_\Gamma f$ if $e \cup f$ contains a C_4.

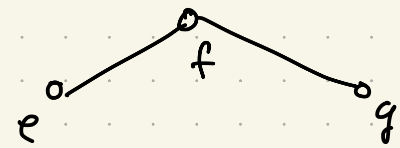


It suffices to find a 'nice' P_3 in Γ

G



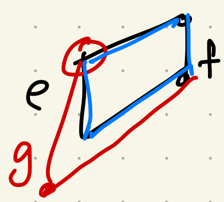
Γ i.e. $e \cap g = \phi$

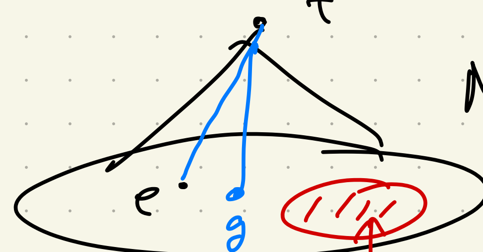


First pick arbitrary $f \in V(\Gamma)$

and e in $N_\Gamma(f)$

BAD



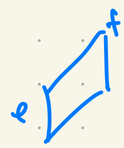


$N_\Gamma(f)$

each BAD g shares an endpt w/ e

$\Rightarrow$ # bad $g \leq 2\Delta(G)$

$\leq 2Knp$



#Bad $\leq 2Knp$

then pick some g (not bad)

For a typical f, $d_\Gamma(f) \geq$ ave $\geq$ # C_4 containing f

$$\geq \frac{\frac{1}{2K} n^5 p^5}{\frac{1}{2} n^2 p} \geq \frac{1}{K} n^3 p^4$$

$$> 2Knp$$

Ex Prove Bondy-Sim. $ex(n, C_{2k}) = O(n^{1+1/k})$ using this method.