Lecture 18

- The missing part of the pf

We shall find distinct vxs $u_1, \dots, u_{t-1} \in A$

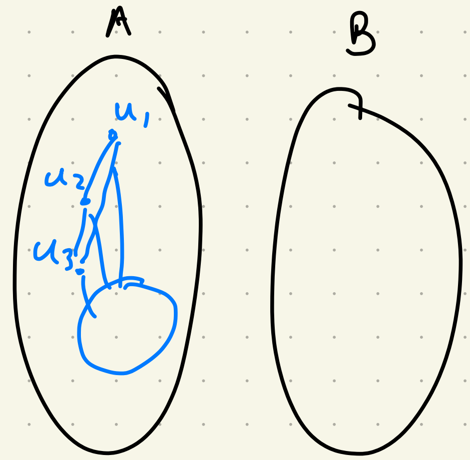
s.t. (i) each $u_i u_j \in W_L$ is a light edge;

(ii) $\forall$ distinct i,j,k, $N^*_G(u_i, u_j, u_k) = \phi$

common neighbrhd

(iii) $\forall\ i \in [t-1]$, $\left|N^*_{W_L}(u_1, \dots, u_i)\right| \geq \left(\frac{\delta^2}{32t^3 n}\right)^i |A|$.

Idea: use locally denseness to find u_i one by one.

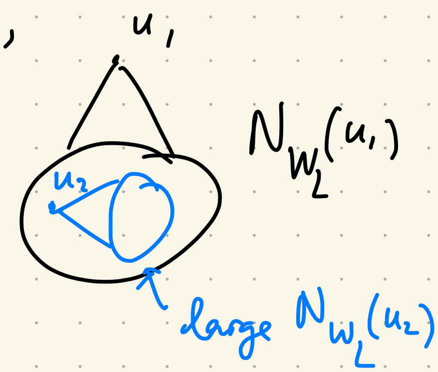


- Take a typical vertex $u_1 \in A$, so u_1 has at least the average # light edges incident to it.

$t=4$

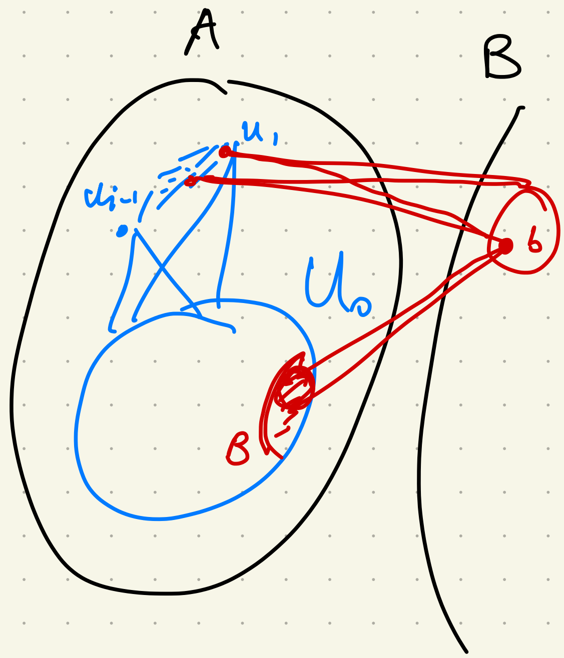
Cor $\Rightarrow$ # light edges incident $u_1 \geq \frac{\delta^2}{8t^3 n} \cdot (|A| - 1) \geq \frac{\delta^2 |A|}{32 t^3 n}$

Suppose now we have found $u_1, \dots, u_{i-1}$, for some $2 \leq i \leq t-1$.



By (iii) $U_0 := N^*_{W_L}(u_1, \dots, u_{i-1})$

$$|U_0| \geq \left(\frac{\delta^2}{32t^3 n}\right)^{i-1} |A|.$$



Let $B \subseteq U_0$ be the set of vxs that have a common neighbr (in G) with $u_{i'}, u_{i''}$ for some $i' \neq i'' \leq i-1$.

- $|B| \leq \binom{i-1}{2} \cdot \binom{t}{2} \cdot K\delta < \frac{1}{2}|U_0|$

as $cn^{\frac{t-2}{2t-3}} \leq \delta \leq n$.

Note that $|U_0 \setminus B| \geq \frac{1}{2}|U_0| \geq \frac{1}{2}\left(\frac{\delta^2}{32t^3n}\right)^{i-1}|A|$

$$\geq \frac{8tn}{\delta}$$

So Cor to $U_0 \setminus B$

$\Rightarrow$ a typical vx there has $\geq \frac{\delta^2}{8t^3n}(|U_0 \setminus B| - 1)$

(u_i)

$$\geq \left(\frac{\delta^2}{32t^3n}\right)^{i}|A|$$

light neighbrs in $U_0 \setminus B$.

☺

Rmk: Dense constructions of $K_t^{(l)}$-free graphs could have important applications to the study of pseudorandom graphs.

- Alon '90s: $\exists$ n-vx
  - Δ-free
  - (edge) density $= \Omega(n^{-1/3})$

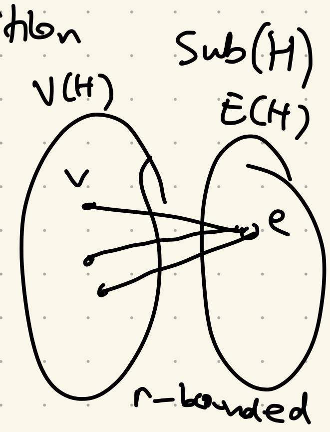
  and
  - pseudorandom.

Much denser than one would get from $G(n,p)$ (density around $n^{-1/2}$).

- Alon-Krivelevich: - n-vx - K_t-free - pseudorandom, density $= \Omega(n^{-1/(t-2)})$

- Conlon: alternative dense pseudorandom Δ-free graphs by 'unsubdividing' C_6-free graphs.

Def Given a hypergraph H, its subdivision is a bip. graph $Sub(H)$ on bipartition $V(H) \cup E(H)$ and $v \in V(H)$, $e \in E(H)$ $v \sim e$ iff $v \in e$.



Conj (Conlon-Lee) $\forall\ r \geq 3$, $\forall$ r-graph H $\exists\ c > 0$ s.t.

$$ex(n, \mathrm{sub}(H)) \leq n^{2 - 1/r - c}$$

Known

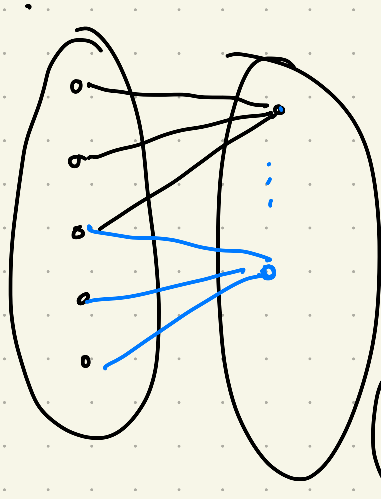
- H : r-partite r-unif [C-L]
- $H = K_{r+1}^{(r)}$ via reduction thm of Erdős-Sim. and the result on Q_3.

$$ex(n, \mathrm{sub}(K_{r+1}^{(r)})) = O\left(n^{2 - \frac{2}{2r-1}}\right)$$

Testing cases

- H linear hypergraph
- $K_{r+2}^{(r)}$

$r = 3$



$\binom{[5]}{3}$

Erdős's Conj $H \begin{cases} r\text{-degenerate} \\ \text{bip.} \end{cases} \Rightarrow ex(n, H) = O(n^{2 - 1/r})$

Alon-Krivelevich-Sudakov: $O(n^{2 - 1/4r})$

§ Sidorenko's conj.

Sidorenko's conj. is a conj about graph hom. ineq. relating subgraphs densities and edge density

<u>Recall</u> A homomorphism from H to G is a map $\varphi : V(H) \to V(G)$ preserving adjacencies, i.e.

$$\forall uv \in E(H) \Rightarrow \varphi(u)\varphi(v) \in E(G).$$

- $H \xrightarrow{\text{hom}} K_r$ means H has chr. # $\leq r$.

<u>Notation</u> $\mathrm{Hom}(H, G)$ = set of all hom. from H to G.

$$\mathrm{hom}(H,G) = |\mathrm{Hom}(H,G)|.$$

<u>Def</u>. The hom. density of H in G is the fraction of maps that are hom., i.e.

$$t(H,G) = \frac{\mathrm{hom}(H,G)}{|V(G)|^{|V(H)|}}$$

$$t(K_2, G) = \text{edge density}.$$

Conj (Sidorenko) Let H be a bip. graph.

Then for any G,

$$t(H,G) \geq t(K_2,G)^{e(H)} \quad \cdots\cdots \quad (*)$$

If G has n vxs and $p = t(K_2, G) = \frac{2e(G)}{n^2}$

$$(*) \iff \quad \hom(H,G) \geq n^{v(H)} \cdot p^{e(H)} .$$

The RHS above is the expected # $H \xrightarrow{\text{hom.}} G(n,p)$

So Sidorenko's conj. states that given edge density p, the binomial random graph $G(n,p)$ min. hom. density of any bip. H.

Known

- Sidorenko 91: H trees, even cycles, $K_{s,t}$
- Hatami: hypercube
- Conlon-Fox-Sudakov: H w./ one vx completely joined the other part.
- Li-Szegedy (entropy)

- blow ups $\leftarrow$ (Conlon-Lee)
- finite reflection gp ....

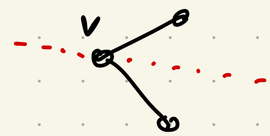
Rmk: H bip is necessary.

Consider $H = K_3$, $G = K_2$

$$0 = t(H,G) \geq t(K_2,G)^{e(H)} = \frac{1}{8}$$

$$t(K_2,K_2) = 1/2$$

Warm up · P_3 case.



NTS $\forall G$: $\hom(P_3, G) \geq n^3 p^2$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

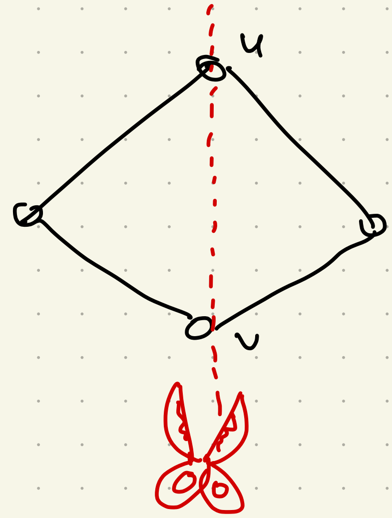
$$\hom(P_3, G) = \sum_{v \in V(G)} d(v)^2$$

$$\overset{C\text{-}S}{\geq} \frac{1}{n}\Big(\underbrace{\sum_{v\in V(G)} d(v)}_{2e(G) = pn^2}\Big)^2$$

Recall $p = \frac{2e(G)}{n^2}$

$$= p^2 n^3$$

- C_4 case

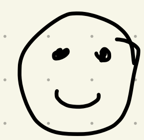


<u>NTS</u> $\hom(C_4, G) \geq n^4 p^4$

$$\hom(C_4, G) = \sum_{u,v \in V(G)} d(u,v)^2$$

$$\overset{C\text{-}S}{\geq} \frac{1}{n^2} \Big(\underbrace{\sum_{u,v \in V(G)} d(u,v)}_{= \hom(P_3, G)} \Big)^2 = \frac{1}{n^2} \hom(P_3, G)^2$$

$$\geq \frac{1}{n^2} \left(n^3 p^2\right)^2 = n^4 p^4$$



- <u>C_{2k} case</u> : $\hom(C_{2k}, G) \geq n^{2k} p^{2k}$

Let A be the adj matrix of G $n \times n$. (real & symm) and so A has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

<u>Prop</u> $\hom(C_{2k}, G) = \#$ closed $2k$-walks in G

$$= \operatorname{tr}(A^{2k})$$

$$= \sum_{i=1}^{n} \lambda_i^{2k} .$$

By Courant-Fischer, $\lambda_1 \geq d(G)$ (average deg)

$$\hom(C_{2k}, G) = \sum_{i=1}^{n} \lambda_i^{2k} \geq \lambda_1^{2k} \geq d(G)^{2k}$$

$$\left(d(G) = np = \frac{2e(G)}{n} \Rightarrow \right) \geq n^{2k} p^{2k}.$$

Open case: Mobius band $K_{5,5} \setminus C_{10}$

(remove edges of C_{10} from $K_{5,5}$.)