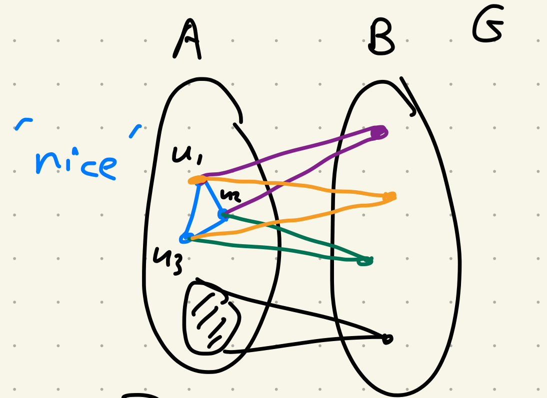
Lecture 17

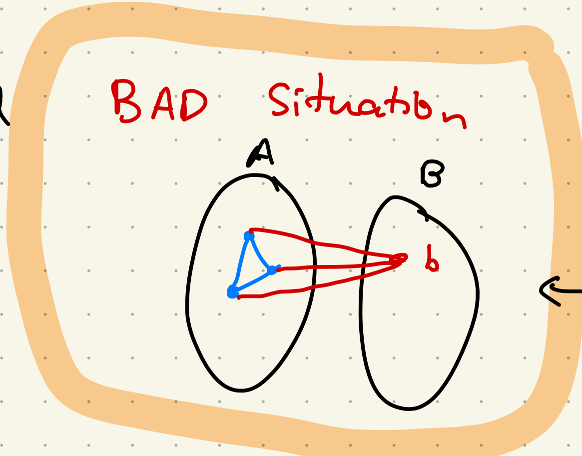
Attempt $t=3$ case : $ex(n, K_3^{(1)}) = ex(n, C_6) \le Cn^{4/3}$.

- Consider the square Γ on A

  If $\exists$ 'nice' Δ, then ☺



Need to avoid



Γ (sufficient cond. for finding 1-subd.)

p = edge density of Γ $\approx \frac{d^2}{n}$

G n-vx, $d = Cn^{1/3}$-reg $\Rightarrow C_6 \subseteq G$.

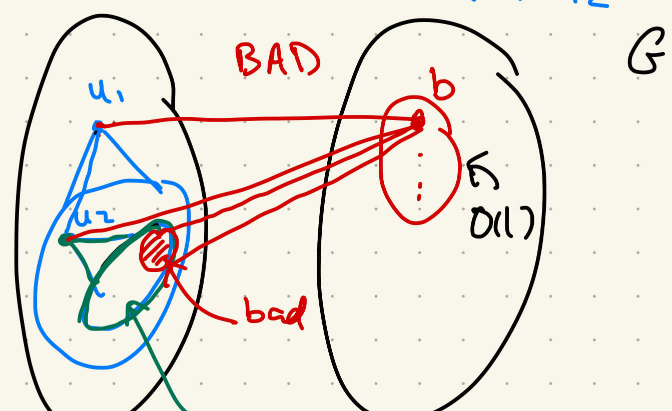
- u_1 typical

$d_\Gamma(u_1) \gtrsim d^2 \approx n^{2/3}$

- u_2 typical

$d_\Gamma(u_1,u_2) \approx \frac{d^4}{n} \approx n^{1/3}$

$n/2 = |A|$ $\qquad$ $|B| = n/2$



$$d_\Gamma(u_1,u_2) \approx d_\Gamma(u_1) \cdot \frac{d^2}{n} \approx \frac{d^4}{n} \approx n^{1/3}$$

\# bad vxs to exclude in $N^*_\Gamma(u_1,u_2)$

$$\le d_G(u_1,u_2) \cdot \Delta(G) = O(d \cdot d_G(u_1,u_2))$$

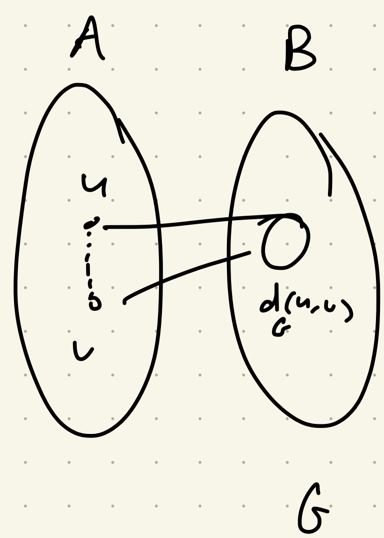
Thus, if $d_G(u_1,u_2) = O(1)$, then we can greedily find u_3

This suggests that it is useful consider the square Γ as a weighted where weights of u_1, u_2 is $d_G(u_1, u_2)$ and we want to weight of $u_1 u_2$ to be light. (i.e. $O(1)$)

Goal upp bd on $ex(n, K_t^{(l)})$

<u>Def</u>: Given a bip. G on $A \cup B$, the weighted square of G is a weighted graph W on vx set A where $\forall uv \in \binom{A}{2}$

$$W(uv) = d_G(u,v)$$

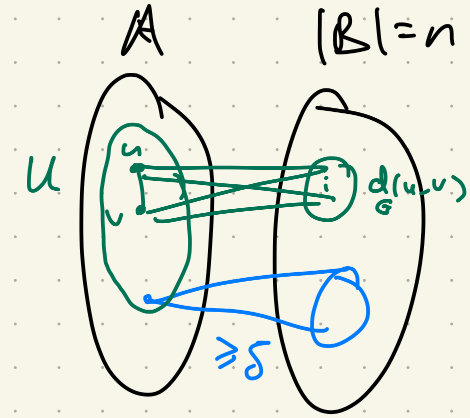


- For $U \subseteq A$, let
$$W(U) := \sum_{uv \in \binom{U}{2}} W(uv)$$

- Call an edge uv in W $\begin{cases} \text{heavy} & \text{if } W(uv) \geq \binom{t}{2} \\ \text{light} & \text{if } 1 \leq W(u,v) < \binom{t}{2} \end{cases}$

Nice property of the square graphs

- By convexity $\Longrightarrow$ the square graphs are locally dense



Lem1
- bip G on $A \cup B$
- $|B| = n$
- min deg $\geq \delta$ on vxs in A

$\Rightarrow \forall\, U \subseteq A$ w/ $|U| \geq \frac{2n}{\delta}$,

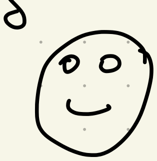
$$W(U) \geq \frac{\delta^2}{2n}\binom{|U|}{2}$$

As each vx $b \in B$ contributes $\binom{d_G(b,U)}{2}$ weight to U, we have

Pf: $\Rightarrow$ $W(U) = \sum_{uv \in \binom{U}{2}} W(uv) = \sum_{b \in B} \binom{d_G(b,U)}{2}$

$$\overset{\text{Convexity}}{\geq} n \cdot \binom{\underbrace{\sum_{b \in B} d_G(b,U)}_{= \text{deg sum of } U} / n}{2} = n \cdot \binom{\frac{\sum_{u \in U} d_G(u)}{n}}{2}$$

$$\geq n \cdot \binom{\delta |U| / n}{2} \geq \frac{\delta^2}{2n}\binom{|U|}{2}$$ as $|U| \geq \frac{2n}{\delta}$



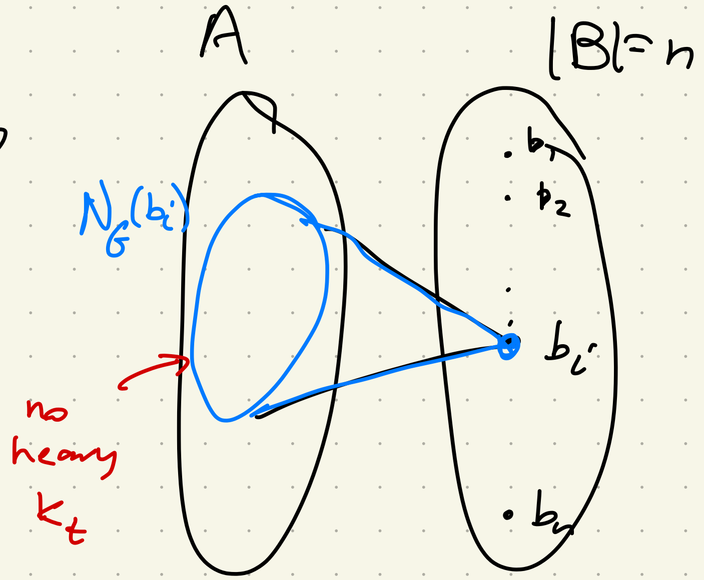
- No heavy K_t in W $\Rightarrow$ positive fraction of edges are light edges

Def: Denote by $W_L \subseteq W$ the spanning subgraph of W consisting of all light edges

Lem 2

- bip. G on $A \cup B$
- $|B| = n$
- $K_t^{(1)}$-free
- $W(A) \geq 8t^2 n$

$\Longrightarrow e(W_L) \geq \dfrac{W(A)}{4t^3}$ # light edges in W



Pf: G is $K_t^{(1)}$-free $\Rightarrow$ W has no heavy K_t

In particular, $\forall\, b_i \in B$

Turán's thm

$\Rightarrow$ # light edges in $N_G(b_i)$ is

$$\geq (t-1)\binom{\frac{d_G(b_i)}{t-1}}{2} \geq \frac{d_G(b_i)^2}{4(t-1)} \quad (\heartsuit)$$

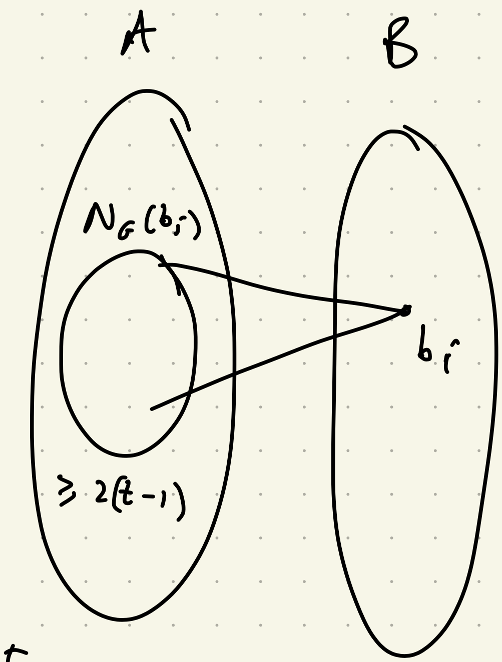
By double counting

$$W(A) = \sum_{i=1}^{n} \binom{d_G(b_i)}{2} = \underbrace{\sum_{i:\, d_G(b_i) \geq 2(t-1)} \binom{d_G(b_i)}{2}}_{\mathrm{I}} + \underbrace{\sum_{i:\, d_G(b_i) < 2(t-1)} (\cdots)}_{\mathrm{II}}$$

$$\mathrm{II} < 4t^2 n \leq \frac{W(A)}{2}$$

$$\Rightarrow \mathrm{I} \geq \frac{W(A)}{2}$$

$$\sum_{i:\, d_G(b_i) \geq 2(t-1)} \binom{d_G(b_i)}{2} \geq \frac{W(A)}{2} \quad \cdots\cdots (\omega)$$



- As every light edge by defn lies in $\leq \binom{t}{2}$ of the sets $N_G(b_i)$

$\Rightarrow$ # total light edges is at least

$$(\heartsuit) \geq \frac{1}{\binom{t}{2}} \sum_{i:\, d_G(b_i) \geq 2(t-1)} \frac{d_G(b_i)^2}{4(t-1)} \overset{(\omega)}{\geq} \frac{W(A)}{4t^3}$$

<u>Cor3</u>
- bip. G on $A \cup B$
- $|B| = n$
- $K_t^{(1)}$-free
- min deg $\geq \delta$ on vxs in A

$\Rightarrow$ $\forall\, U \subseteq A$ w/ $|U| \geq 8tn/\delta$

$$e(W_L[U]) \geq \frac{\delta^2}{8t^3 n} \binom{|U|}{2}$$

<u>Pf</u>
- <u>Lem 1</u> $\Rightarrow$ $W(U) \geq \frac{\delta^2}{2n} \binom{|U|}{2} \geq 8t^2 n$
- Apply Lem 2 on $G[U \cup B]$.

Using standard regularisation lem (like the Erdős-Simonovits one) we may work w/ balanced almost regular graphs.

<u>Thm</u> (Janzer) For every $K \geq 1$ and $t \geq 3$ $\exists\ c = c(K,t)$ and $n_0 = n_0(K,t)$ s.t. TFH $\forall n \geq n_0$.

$\forall$ • bip G on $A \cup B$

• $|B| = n$, $\frac{n}{2} \leq |A| \leq 2n$

• K-almost regular

$\delta \leq d_G(v) \leq K\delta$

• $\delta \geq c n^{\frac{t-2}{2t-3}}$

$$\Longrightarrow \exists\ K_t^{(1)} \subseteq G$$

<u>Pf</u>: We shall find distinct vxs $u_1, \dots, u_{t-1} \in A$ s.t. (i) each $u_i u_j \in W_L$ is a light edge;

(ii) $\forall$ distinct i,j,k, $N^*(u_i,u_j,u_k) = \phi$

common neighbrhd

(iii) $\forall\ i \in [t-1]$, $|N^*_{W_L}(u_1,\dots,u_i)| \geq \left(\frac{\delta^2}{32t^3 n}\right)^i |A|$.

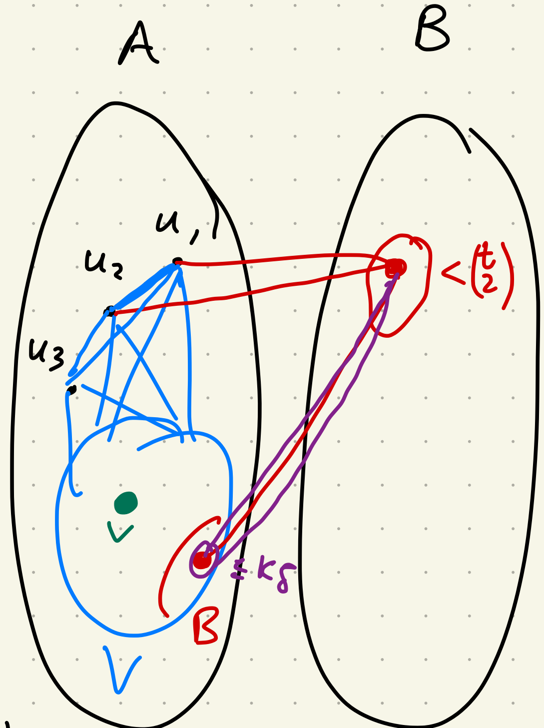
<u>Rmk</u>: $\Pr(u \sim v \text{ in } W) \approx \delta^2/n$

So $\Omega\left(\left(\frac{\delta^2}{n}\right)^i |A|\right)$ is what we would expect.

Suppose we found such $u_1, \dots, u_{t-1} \in A$.

Write $V = N^*_{W_L}(u_1, \dots, u_{i-1})$

- By (iii), $|V| \geq \left(\frac{\delta^2}{32t^3 n}\right)^{t-1} |A|$



Let B be the set of (bad) v's in V that has a common neighbr (in G) w/ some pair $u_i u_j$.

calculation

$$|B| \leq \binom{t-1}{2} \binom{t}{2} \cdot k\delta < |V|$$

\# pairs $u_i u_j$ $\qquad d_G(u_i, u_j)$

$t = 4$

Pick $v \in V \setminus B$, $\quad u_1, \dots, u_{t-1}, v \Rightarrow K_t^{(1)}$.

- $u_1, u_2, \dots, u_{t-1} \Rightarrow K_{t-1}^{(1)}$ by (ii)
- by choice of $v \notin B$ and (ii)