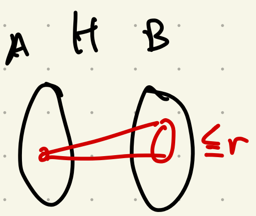
Lecture 16

§ Random zooming

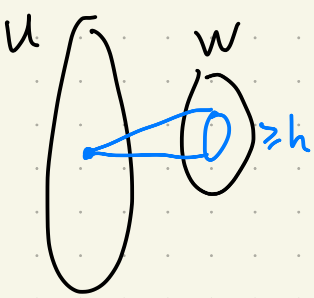
<u>Idea</u>: Randomly zoom into an asymmetric subgraph and do the embedding using the created asymmetry.

Prop 1

- $H = (A \cup B, E(H))$
  - h-vertex
  - vxs in A have deg $\leq r$



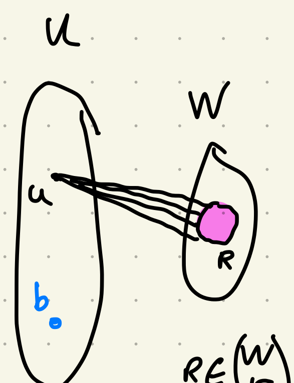
- $G = (U \cup W, E(G))$
  - vxs in U have deg $\geq h$
  - $|U| > h\binom{|W|}{r}$



$\Rightarrow H \subseteq G$

<u>Pf</u>: Take a maximal partial coloring $\varphi : U \longrightarrow \binom{W}{r}$ s.t.

- if $\varphi(u) = R$, then $R \subseteq N(u)$;
- every color class $\varphi^{-1}(R)$ has size $\leq h$;
- all color classes are pairwise disjoint.



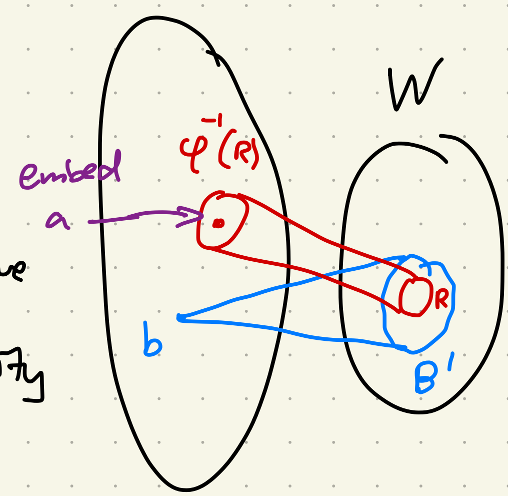
As $|U| > h\binom{|W|}{r}$, some vertex $b \in U$ is uncolored.

$R \in \binom{W}{r}$

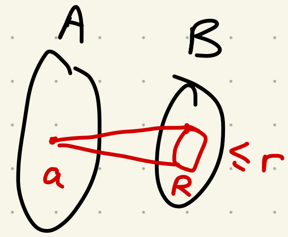
Consider an h-subset $B' \subseteq N(b)$.

Note that, for each color $R \in \binom{B'}{r}$,

$|\varphi^{-1}(R)| = h$, for otherwise we could have color the vx b w/ color R, ↯ maximality of φ.



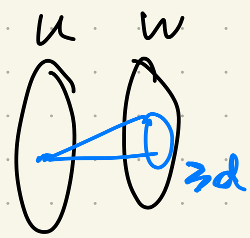
As all color classes are pairwise disjoint, we can greedily embed H as follows.



- Let f be an embedding $B \xrightarrow{\text{into}} B'$
- $\forall a \in A$, let $R \in \binom{B'}{r}$ be an r-set containing $N_H(a) \longrightarrow$ embed a to an unused vx in $\varphi^{-1}(R)$

Thm (GF-H-L.-P-W: Disjoint isomorphic balanced clique subdivision)

Let $d \geq \max\{40, 2h\}$ and G be a bip. graph on partite sets $U \cup W$ s.t. every vx in U has deg $\geq d$ in W.



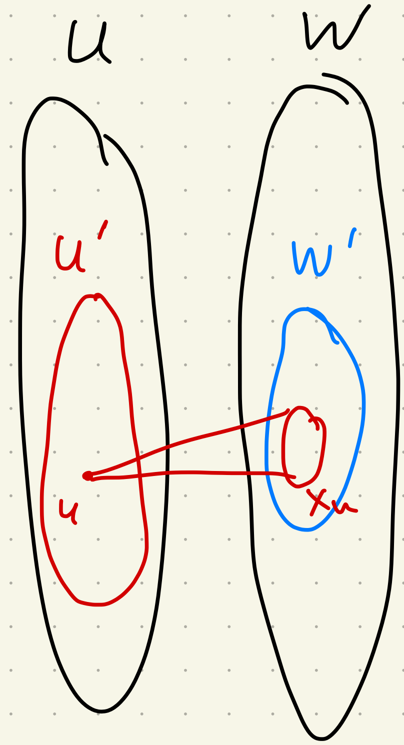
If ♡ $\underbrace{\frac{1}{2|W|} \cdot \left(\frac{|U|}{4h}\right)^{1/r}}_{p} \cdot \frac{d}{2} \geq \max\{20, h\}$

$\Rightarrow$ then G contains any h-vx bip. H w/ max. deg r on one part.

Pf: • We shall find appropriate subsets $U' \subseteq U$, $W' \subseteq W$ to invoke the previous prop.

• Let $p := \frac{1}{2|W|}\left(\frac{|U|}{4h}\right)^{1/r}$. If $|U| > 4h(2|W|)^r$, then replace U be a subset of size exactly $4h(2|W|)^r$, ♡ still holds and $p \leq 1$.

- Let $W' \subseteq W$ be a random subset of W where each vx of W is included w/ prob. p indep. of others.



- For $u \in U$, let $X_u = d_G(u, W')$

sum of indep. Bern. random var.
$$= \sum_{w \in N(u)} \mathbb{1}_{\{w \in W'\}}$$

$$\mathbb{E} X_u = p \cdot d(u) \geq pd$$

- $\left\{ \begin{array}{l} \text{low-tail Chernoff} \\ pd/12 \geq 2 \end{array} \right.$ $\Rightarrow \Pr(X_u < pd/2) \leq \exp(-pd/12) \leq 1/4$ for each $u \in U$.

- Let $U' = \{u \in U : X_u \geq pd/2\}$

$$\Rightarrow \mathbb{E}|U'| = |U| \cdot \Pr(X_u \geq pd/2) \geq \frac{3|U|}{4}$$

- Suppose $q = \Pr(|U'| < |U|/4) > 1/2$

$$\Rightarrow \mathbb{E}|U'| \leq q \cdot |U|/4 + (1-q)|U| = |U|(1 - 3q/4)$$
$$\leq 5|U|/8 \quad ↯$$

So $\Pr(|U'| < |U|/4) \leq 1/2$

- Now, as $|W| \geq d \Rightarrow \mathbb{E}|W'| = p|W| \geq pd \geq 40$

upper tail Chernoff $\Rightarrow$ $\Pr(|W'| > 2p|W|) \leq \exp(-p|W|/3)$

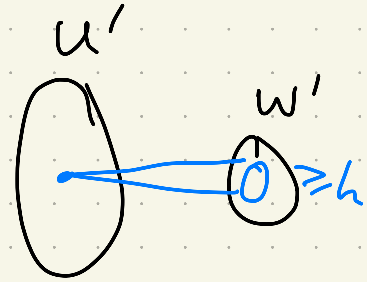
$\leq 1/4$

Thus, $\Pr(|W'| > 2p|W|) + \Pr(|U'| < |U|/4) < 1$

Therefore, $\exists$ a choice of W' s.t.

$|U'| \geq |U|/4$

$|W'| \leq 2p|W| = \left(\frac{|U|}{4h}\right)^{1/r}$



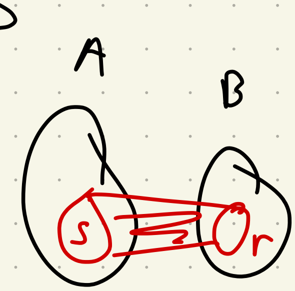
- Choice of U' $\Rightarrow$ vxs in U' have deg $\geq pd/2 \geq h$ in W'.

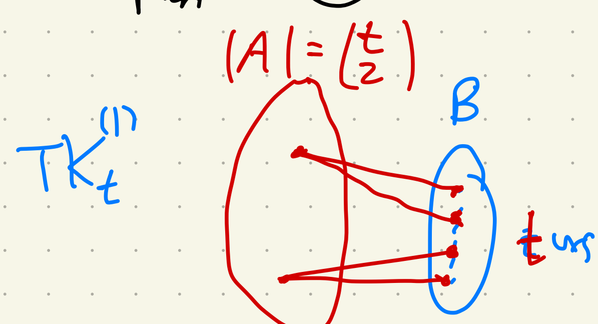
$|U'| \geq |U|/4 \overset{(\heartsuit)}{\geq} h|W'|^r > h\binom{|W'|}{r}$.

Prop $\rightarrow G(U', W')$

<u>Exercise</u> If H is a bip. on $A \cup B$

- vxs in A have deg $\leq r$, and
- H has no $K_{s,r}$ w/ the s-vx part in A.





Improve
- Prop 1 to $|U| > s\binom{|W|}{r}$
- Thm to ($\heartsuit$) $\frac{1}{2|W|}\left(\frac{|U|}{2s}\right)^{1/r}\cdot\frac{d}{2} \geq \max\{20, h\}$.

Note that 1-subdivision of K_t has no $K_{2,2}$.

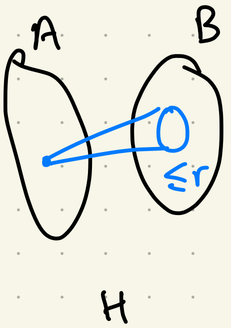
Exercise $\forall$ n-vx G w/ cn^2 edges

$\Rightarrow \exists\ TK_t^{(1)}$ w/ $t \geq \Omega(c\sqrt{n})$.

Exercise: Thm $\Rightarrow$ F-AKS.

§ Improving F-AKS

[FAKS] H one part deg $\leq r$ $\Rightarrow$ $ex(n, H) = O(n^{2-1/r})$

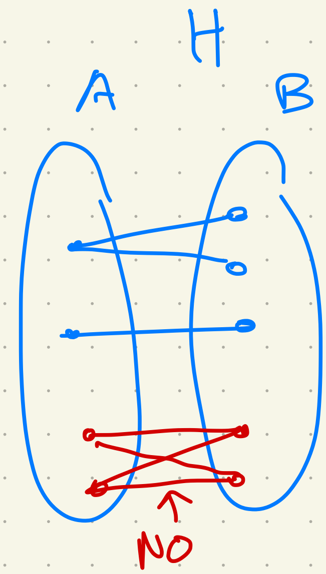


Conlon-Lee conj FAKS only tight when H contains $K_{r,r}$. i.e.

$\forall$ bip H
- r-bounded
- $K_{r,r}$-free

$\Rightarrow ex(n, H) = O(n^{2-1/r-c_H})$

Still open for any $r \geq 3$.



$r=2$ Case every such H ($K_{2,2}$-free, 2-bdd) is a subgraph of 1-subdivision of K_t ($t \geq |B|$).

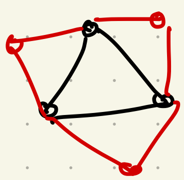
- $r=2$ case proved by Conlon–Lee.

We present a better bd by Janzer.

Thm (Janzer) $\forall t \geq 3$, $ex(n, K_t^{(1)}) = O(n^{\frac{3}{2} - \frac{1}{4t-6}})$

Rmk
- tight $t=3$ $ex(n, C_6) = \Theta(n^{4/3})$

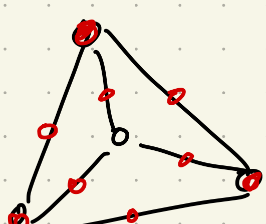
C_6 $K_3^{(1)} = C_6$



- By prob. constr. $ex(n, K_t^{(1)}) = \Omega\left(n^{\frac{3}{2} - \frac{t - 3/2}{t^2 - t - 1}}\right)$

For example, $t=4$:

$$\Omega(n^{\frac{3}{2} - \frac{5}{22}}) \leq ex(n, K_4^{(1)}) \leq O(n^{\frac{3}{2} - \frac{1}{10}}) = O(n^{7/5})$$



polyn.

Q: Better construction for $K_4^{(1)}$ than 1st moment deletion.