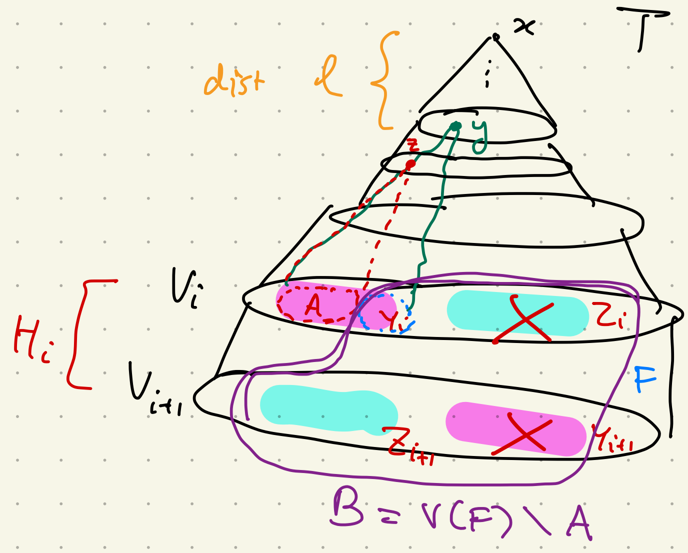
Lecture 15

- More details on missing cases

Θ_k-graph F w/ bipartition $Y \cup Z$



- Θ_k-graph F is bipartite $\geq 2k$

and $\delta(F) \geq 2$, connected!

If, wlog, $Y \cap V_i \neq \phi$

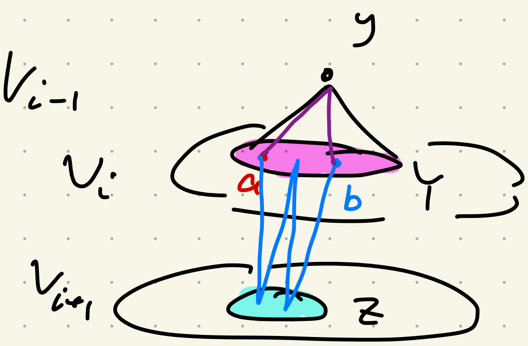
$\Rightarrow$

we must have $Y \subseteq V_i$, $Z \subseteq V_{i+1}$

- Define y as before. Case 1; $y \notin V_{i-1}$, define z

as before, ... do the same pf.

Case 2 (Missing case) if $y \in V_{i-1}$



then take an arbitrary $a \in Y$ and let

$A = \{a\}$. and $B = F \setminus A$

Again $\delta(F) \geq 2 \Rightarrow (A, B)$ is NOT a bip. of F

(as B contains edges)

$\xRightarrow{\text{Lem}}$ $\exists$ A,B-path P of length $2k-2$ (even)

P has endpts a and b both in Y.

$P + ay + by \Rightarrow C_{2k}$ ↯.

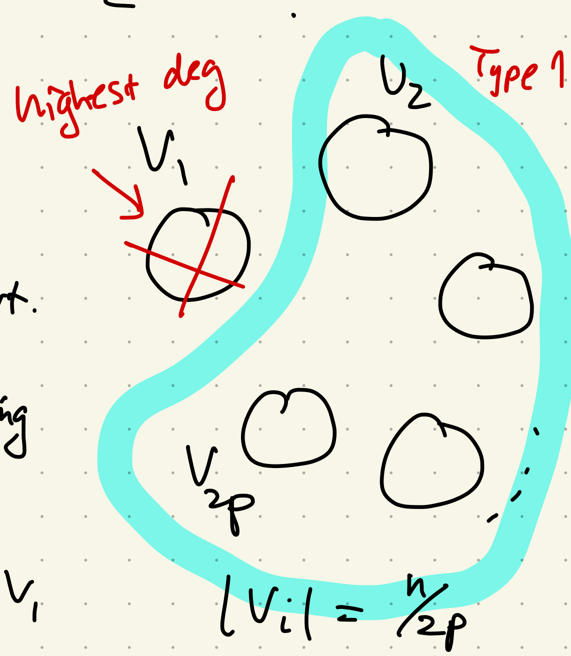
§ Regularisation.

A lem of Erdős and Simonovits allows us to work with almost regular graphs for bipartite Turán problem.

We say a graph G is **K-almost regular** if

$$\Delta(G) \le K \cdot \delta(G)$$

Lem (E-Sim.) Let $0 < \varepsilon < 1$, $c > 0$ and n be suff. large. Let G be an n-vertex graph with $e(G) \geq c \cdot n^{1+\varepsilon}$. $\Rightarrow$ Then G contains an m-vertex K-almost regular subgraph G' w./ $m \geq n^{\frac{\varepsilon-\varepsilon^2}{4+4\varepsilon}}$ and $e(G') \geq \frac{2c}{5} m^{1+\varepsilon}$, where $K = 20 \cdot 2^{\frac{1}{\varepsilon^2}+1}$.



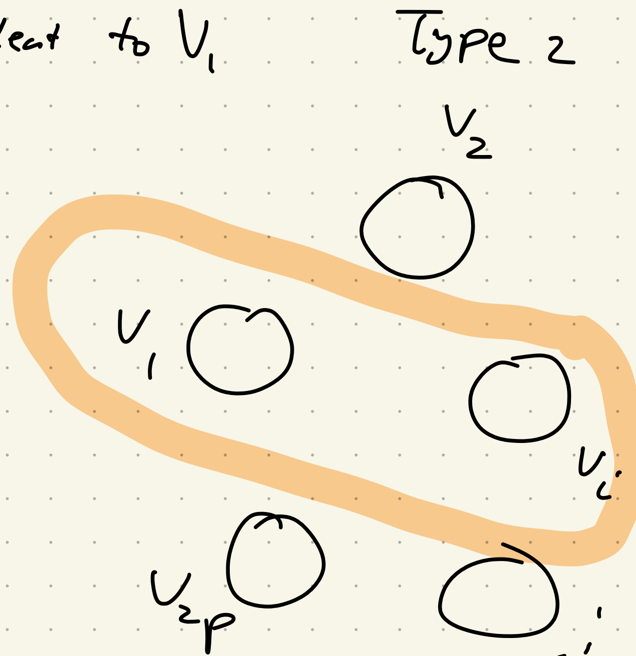
Pf (Sketch, details = exercise)

Let $p = 2^{\frac{1}{\varepsilon^2}+1} = \frac{K}{20}$ and take a $2p$-equipart. $V_1 \cup \dots \cup V_{2p}$ of $V(G)$ with V_1 containing the highest deg. vxs.

- G type 1: if $\le$ half edges incident to V_1
  - delete V_1
  - repeated delete vxs of deg $< \frac{c}{10} n^{\varepsilon}$ until no such vxs

$\nexists$ exist $\Rightarrow$ desired G'.

- G type 2 : if $\geq$ half edges incident to V_1



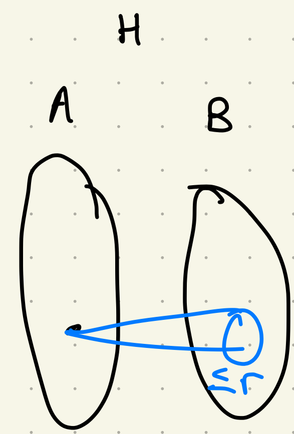
- Pigeonhole $\Rightarrow \exists\, V_i$ s.t.

$$e_G(V_1, V_i) \geq \frac{1}{2p} e(G)$$

- iterate the same analysis on $G[V_1, V_i]$

It can be shown that this process terminates at a large type 1 subgraph.

§ Bip. graphs w./ bdd deg.

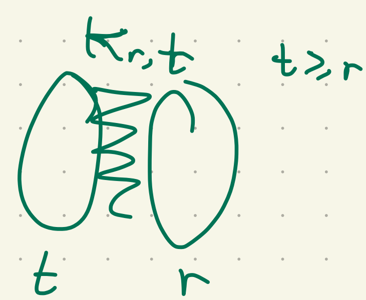


<u>Thm</u> (Füredi , Alon-Krivelevich-Sudakov)

Let H be a bip. graph w./ bipartition $A \cup B$ where each vx in A has deg $\leq r$ in B. Then $\exists$ $c = c(H)$ s.t.

$$ex(n, H) \leq c \cdot n^{2 - 1/r}.$$

<u>Rmk</u> This thm generalise K-S-T.



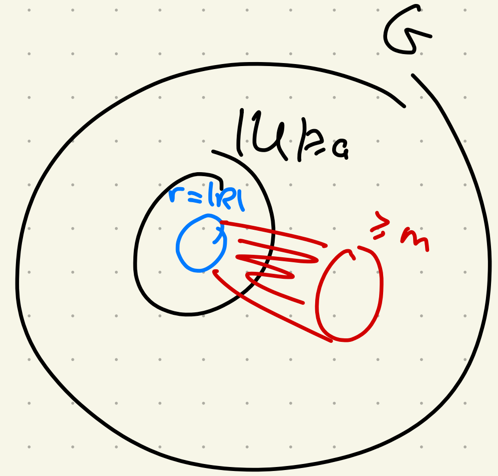
§ 1st pf via dependent random choice lem (DRC)

The following basic lem finds in a dense graph a large set U s.t. all small subsets of U have many common neighbors.

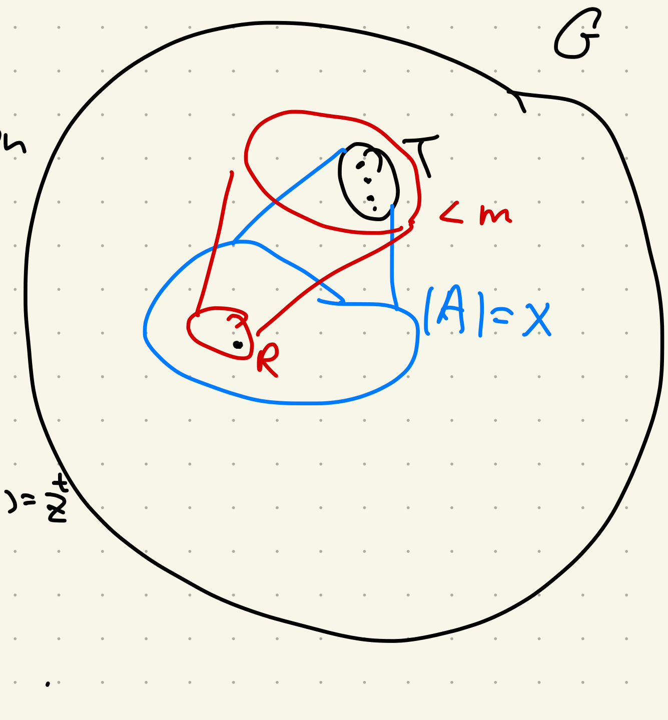
Lem Let $a, d, m, n, r \in \mathbb{N}$. Let $G = (V, E)$ be an n-vertex graph with average deg d. If $\exists\, t \in \mathbb{N}$ s.t.

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a$$

$\Rightarrow$ then G contains a subset U of size $\geq a$ s.t. every r-subset $R \subseteq U$ has $\geq m$ common neighbors.



Pf: • Pick a set T of t vxs of V uniformly at random w/ repetition



Let $A = N^*(T) = \bigcap_{t \in T} N(t)$ and let $X = |A|$.

• Linearity of $\mathbb{E}$

convexity of $f(z) = z^t$

$$\Rightarrow \mathbb{E}X = \sum_{v \in V}\left(\frac{d(v)}{n}\right)^t \geq \frac{d^t}{n^{t-1}}.$$

- Let $Y = \#$ r-sets $R \subseteq A$ w/ $|N^*(R)| < m$.

For a such set R w/ $< m$ common neighbors,

$$\mathbb{P}(R \subseteq A) = \mathbb{P}(T \in N^*(R)) = \left(\frac{|N^*(R)|}{n}\right)^t < \left(\frac{m}{n}\right)^t$$

$$\Rightarrow \quad \mathbb{E}Y \le \binom{n}{r} \cdot \left(\frac{m}{n}\right)^t$$

$$\Rightarrow \quad \mathbb{E}(X - Y) \geq \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a$$

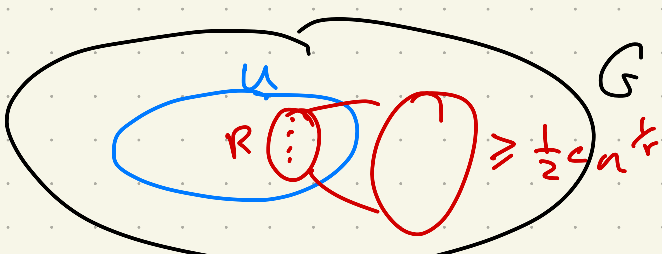
Therefore, $\exists$ a choice of T s.t. $X - Y \geq a$

Delete one vx from each 'bad' R in A

$\Rightarrow$ $|U| \geq X - Y \geq a$ w/ the desired property. ☺

<u>Cor</u> (Exercise) Let G be an n-vx graph w/ $\geq cn^2$ edges and $r \in \mathbb{N}$.

$\Rightarrow$ $\exists$ a subset of vxs U of size $|U| \geq \frac{1}{2}c^r n$ s.t. all r-sets in U have $\geq \frac{1}{2}cn^{1/r}$ common neighbors.

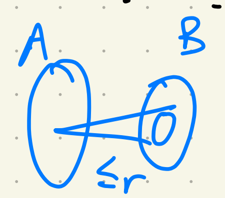


<u>Exercise 1</u> Use the above Cor. to prove

Füredi - Alon - Krivelevich - Sudakov.

• Use dependent random choice lem to prove the following.

<u>Exercise</u> $G \begin{cases} n\text{-vx} \\ cn^2 \text{ edges} \end{cases} \Rightarrow$ 1-subdivision of K_t, where $t \geq c^{3/2} n^{1/2}$.

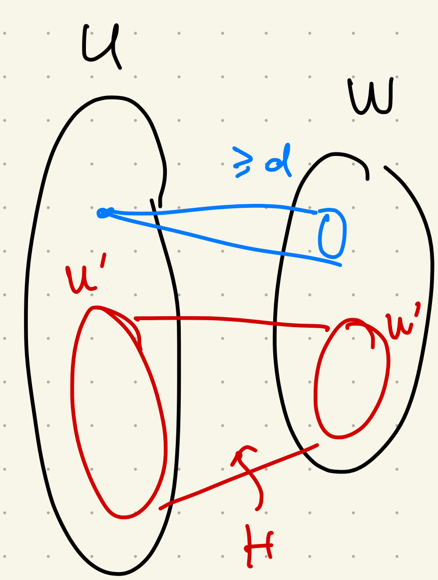


<u>Rmk</u> Fox - Sudakov survey on DRC.

§ 2^{nd} pf of F-AKS via random zooming

Let H be an h-vertex bip. graph w/ bip. $A \cup B$ s.t. every vx in A has deg $\leq r$ in B

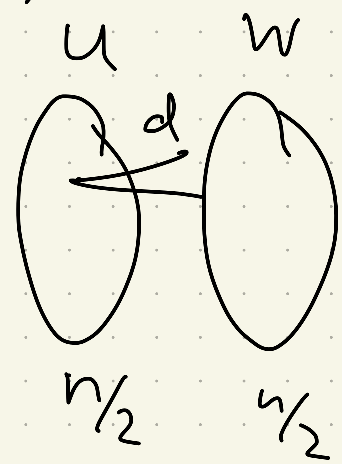
$\boxed{\text{Thm 1}}$ Let $d \geq 40$ and G be a bip. graph w/ bip. $U \cup W$ s.t. every vertex in U has deg d in W.



If $\frac{1}{2|W|} \cdot \left(\frac{|U|}{4h}\right)^{1/r} \cdot \frac{d}{2} \geq 20$

$\Rightarrow$ then $H \subseteq G$

Pf (Thm 1 $\Rightarrow$ F-A-K-S: $ex(n,H) = O(n^{2-\frac{1}{r}})$)



$d = c' n^{1-\frac{1}{r}}$

$$\frac{1}{2|W|} \cdot \left(\frac{|U|}{4h}\right)^{1/r} \cdot d/2$$

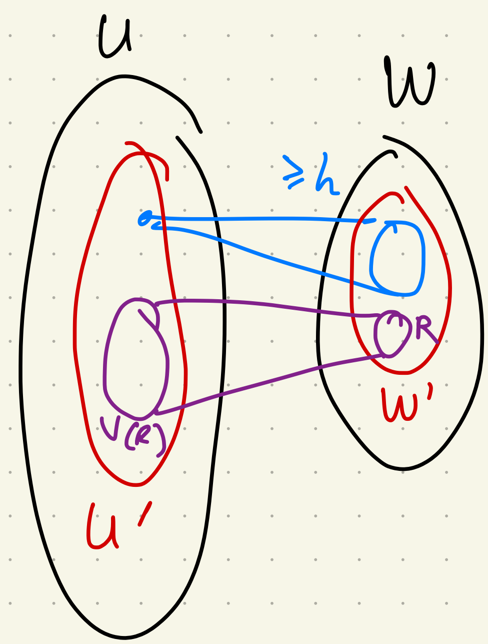
$$= \frac{1}{2 \cdot n/2} \cdot \left(\frac{n}{8h}\right)^{1/r} \frac{c'}{2} n^{1-\frac{1}{r}} \geq 20$$

as long as $c' \gg h, r$.

Exercise: Prove F-A-K-S using Thm 1.

Pf Claim $\exists\ U' \subseteq U,\ W' \subseteq W$ s.t.

- $|U'| \geq h \cdot |W'|^r$ (very skewed)
- every vx in U' has deg $\geq h$ in W'



We shall embed H in $G[U', W']$

- Take a maximal collection

$\mathcal{I} \subseteq \binom{W'}{r}$ for which there is

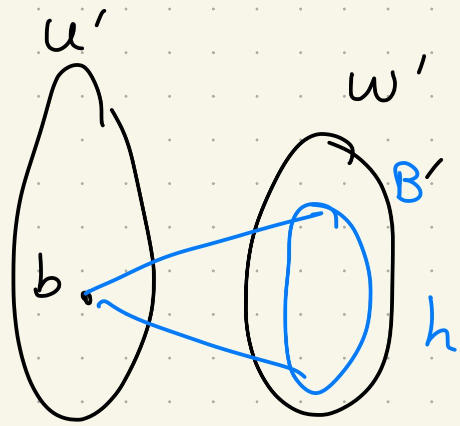
$V(R) \subseteq U'$ for every $R \in \mathcal{I}$ s.t.

$V(R)$ is completely joined to R and $|V(R)| = h$.

- As $|U'| \geq h \cdot |W'|^r > h \cdot \binom{|W'|}{r}$

$\Rightarrow \exists$ a vertex $b \in U'$ s.t. $b \notin V(R)$ for any $R \in \mathcal{I}$.

Take $B' \subseteq N(b, W')$, $|B'| = h$



Embed the B side of H to B'.

greedily embed A using $V(R)$, $R \in \mathcal{I}$.