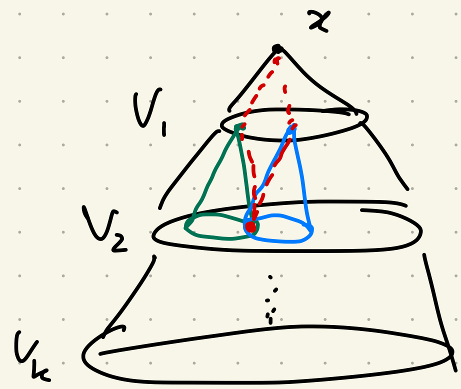
Lecture 14

Thm (Pikhurko) For large n, $ex(n, C_{2k}) \le 4k \cdot n^{1+1/k}$.

To warm up, let us consider forbidding $\mathcal{C}_{2k} := \{C_4, C_6, \dots, C_{2k}\}$

$$ex(n, \mathcal{C}_{2k}) = O(n^{1+1/k})$$

Take a BFS (Breadth First Search) tree T.



$\forall i \le k$

$\mathcal{C}_{2k}$-free $\Rightarrow$ $|V_i| \ge \delta(G)^i$

$$\delta(G)^k \le |V_k| \le |V(G)| = n$$

$$\Rightarrow \delta(G) \le n^{1/k}$$

Rmk: We have to be more careful to implement this BFS approach as now we only forbidding C_{2k}.

Note that C_{2i}, $\forall i<k$, is a (degenerate) homomorphic image of C_{2k}.

The main difficulty in many bip. Turán problem is to control the count of such degenerate homomorphisms.

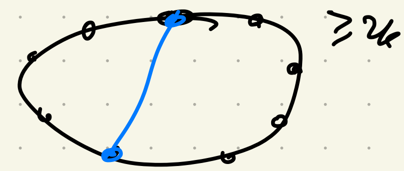
Prop Every graph G contains a subgraph H w/

$$\delta(H) \ge d(G)/2.$$

Prop Every graph G contains a bip. subgraph H w./

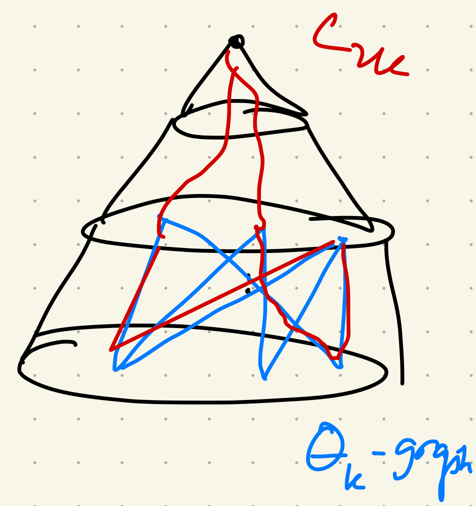
$$e(H) \geq e(G)/2.$$

Def: A Θ_k-graph is a cycle of length $\geq 2k$ with a chord.



Ex Let $k \geq 3$ and H be a bip. graph w./ $d(H) \geq 2k$. Then H contains a Θ_k-graph.

Idea: Consider a BFS tree T.



If between any of the first k pairs of consecutive layers, there is a Θ_k-graph

$\Rightarrow \exists\, C_{2k}$.

Then Ex $\Rightarrow$ each of the first k layers expands by a factor $\Omega\left(\frac{d(G)}{k}\right)$

We need the following Lem. showing that Θ_k-graph contains paths of many varying length.

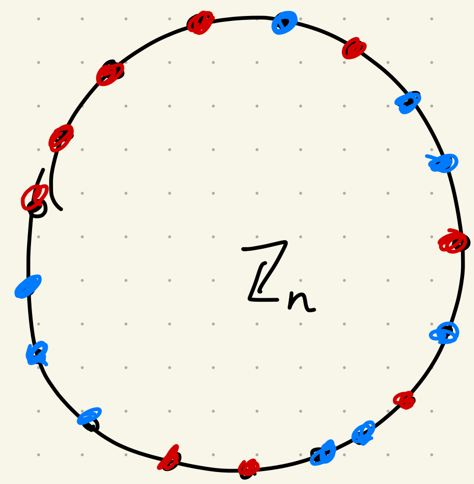
Lem Let F be a Θ_k-graph and $V(F) = A \cup B$ a non-trivial partition $(A, B \neq \phi)$.
If F is not bipartite w./ bipartition $A \cup B$, then there A,B-paths of all lengths less than $|F| = n$.

Pf: Suppose for some $1 \le \ell < n$, there is no A,B-path of length ℓ in F.

We need to prove that F is bipartite w./ bipartition $A \cup B$.

- Identify the spanning cycle in F w./ $\mathbb{Z}_n$

and think of (A, B) partition as a 2-coloring c of $\mathbb{Z}_n$



- Define the set of periods of c

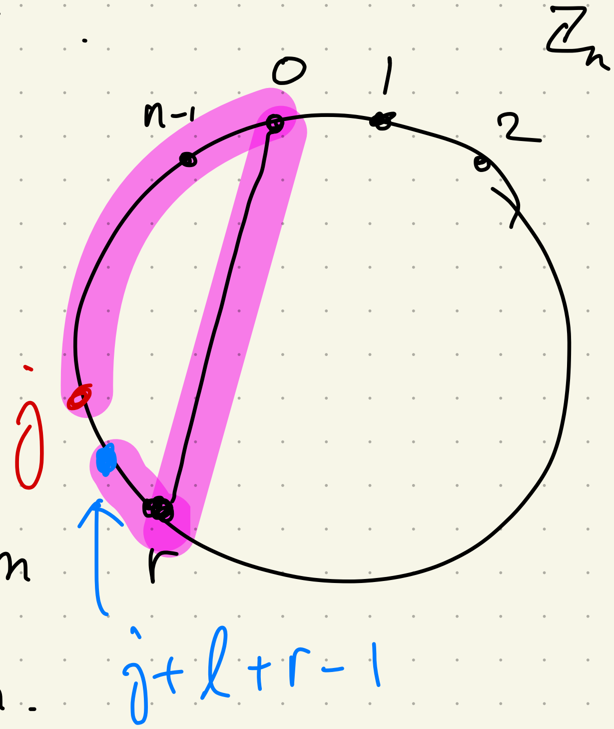
$$P = \{ m \in \mathbb{Z}_n : \forall\, i \in \mathbb{Z}_n,\ c(i) = c(i+m) \}$$

By our assumption, $\ell \in P$.

- $A, B \neq \phi \implies$ the smallest period $m \in P$ divides n, $m \mid n$

$\implies P = \{ mi : i \in \mathbb{Z}_n \}$ ...... Ex

- May assume $m > 2$ for otherwise $A \cup B$ is a bipartition of F.



- Suppose the chord in F connects vxs 0 and r.

- If $n-r \equiv r \equiv 1 \mod m$

$\Rightarrow$ $n = (n-r)+r \equiv 2 \mod m$

contradicting $m > 2$ and $m \mid n$.

- Suppose $r \not\equiv 1 \mod m$, i.e. $r-1 \notin P = \{mi : i \in \mathbb{Z}\}$

Recall $l \in P$, $\Rightarrow \exists$ some $j \in \mathbb{Z}_n$ s.t.

$$c(j) \neq c(j+l+r-1)$$

We may assume $-m < j \leq 0$

$\Rightarrow$ Consider the l-walk

$$j, j+1, \dots, -1, 0, r, r+1, \dots, j+l+r-1$$

is an A,B-walk. We get a contradiction unless $l+r-1 \geq n$. But we can assume $l+r-1 < n$ by substracting multiples of m

while still being a non-period.
The case $n - r \not\equiv 1$ can be handled similarly.

Pf $\left(ex(n, C_{2k}) \leq 4k \cdot n^{1+1/k} \right)$

Let G be an n-vx C_{2k}-free graph w/ $e(G) \geq 4kn^{1+1/k}$.

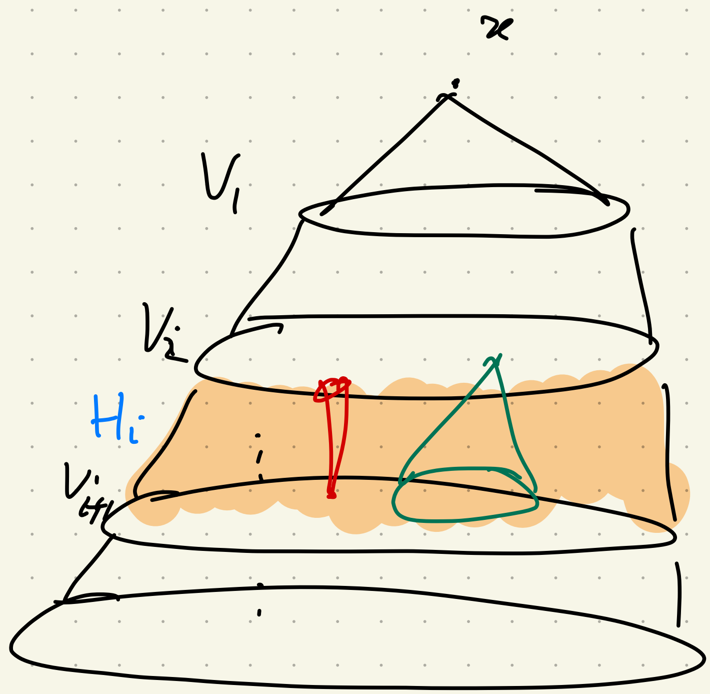
By passing to a subgraph H and lossing a factor of 4

$\Rightarrow$ H bip and $\delta(H) \geq d(G)/4 \geq 2k \cdot n^{1/k}$.

For $i \geq 0$, let

$V_i = N_H^i(x)$

$H_i = H[V_i, V_{i+1}]$



Claim $\forall\, i \leq k-1$,
H_i has no Θ_k-subgraph.

(Claim $\Rightarrow$ Thm) Ex $\Rightarrow$ $d(H_i) \leq 2k - 1$

Iteratively, as $n \gg k$, for $i = 0, 1, 2, \cdots, k-1$,

on average, each vx in V_i sends downward

to V_{i+1} $\geq \delta(H) - O(k)$ edges, while

each vx in V_{i+1} sends back to V_i $\leq 2k-1$ edges.

$\Rightarrow \forall\ 0 \leq i \leq k-1$, the ratio

$$|V_{i+1}| / |V_i| \geq \frac{\delta(H) - O(k)}{2k-1}$$

$$n \geq |V_k| = \frac{|V_k|}{|V_{k-1}|} \cdots \frac{V_1}{1} \geq \left(\frac{\delta(H) - O(k)}{2k-1}\right)^k$$

$\Rightarrow \quad \delta(H) < 2kn^{1+1/k}$ $\lightning$.

<u>Pf</u> (Claim) Suppose H_i contains a Θ_k-graph F.

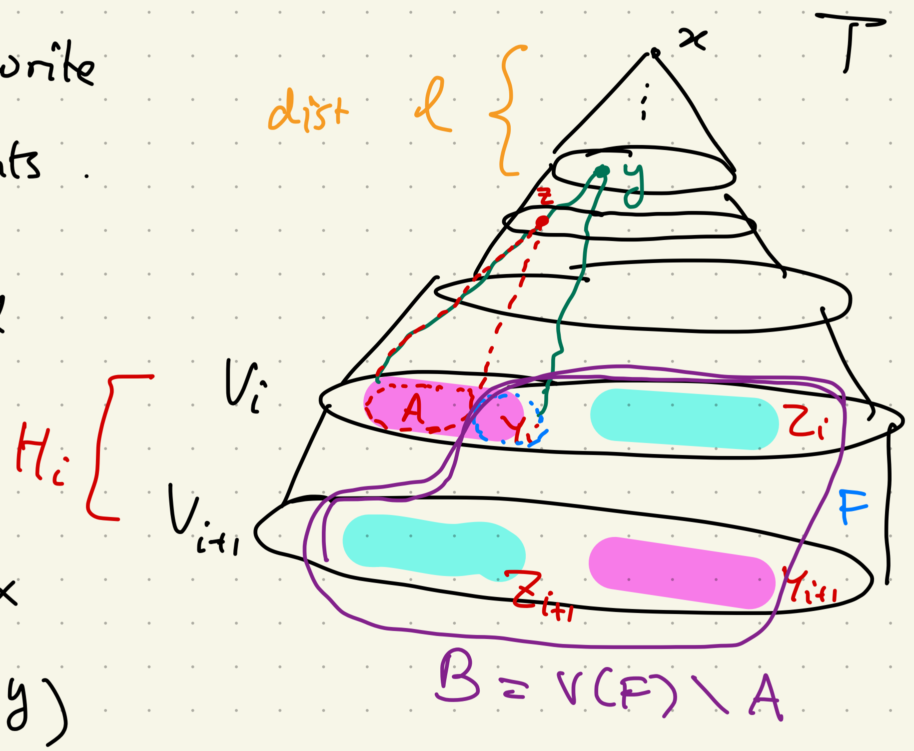
we shall find a copy of C_{2k} for contradiction.

- $F \subseteq H_i$ bipartite, say $Y \cup Z$ is a bipartition of $V(F)$

$Y_i = Y \cap V_i \qquad Z_i = Z \cap V_i$

$Y_{i+1} = Y \cap V_{i+1} \qquad Z_{i+1} = Z \cap V_{i+1}$

For a vx $v \in T$, write $D(v)$ for its descendents.



- Let y be a minimal ancestor of Y_i, that is, y is the farthest vertex from x s.t. $Y_i \subseteq D(y)$

- Take a child z of y s.t. $A = D(z) \cap Y_i \neq \emptyset$

- Minimality of $y \Rightarrow Y_i \setminus A \neq \emptyset$

Consider the nontrivial partition A and $B = V(F) \setminus A$.

- As every vx in F has $\deg \geqslant 2$, there must be edges between $B \cap V_i$ and $B \cap V_{i+1}$

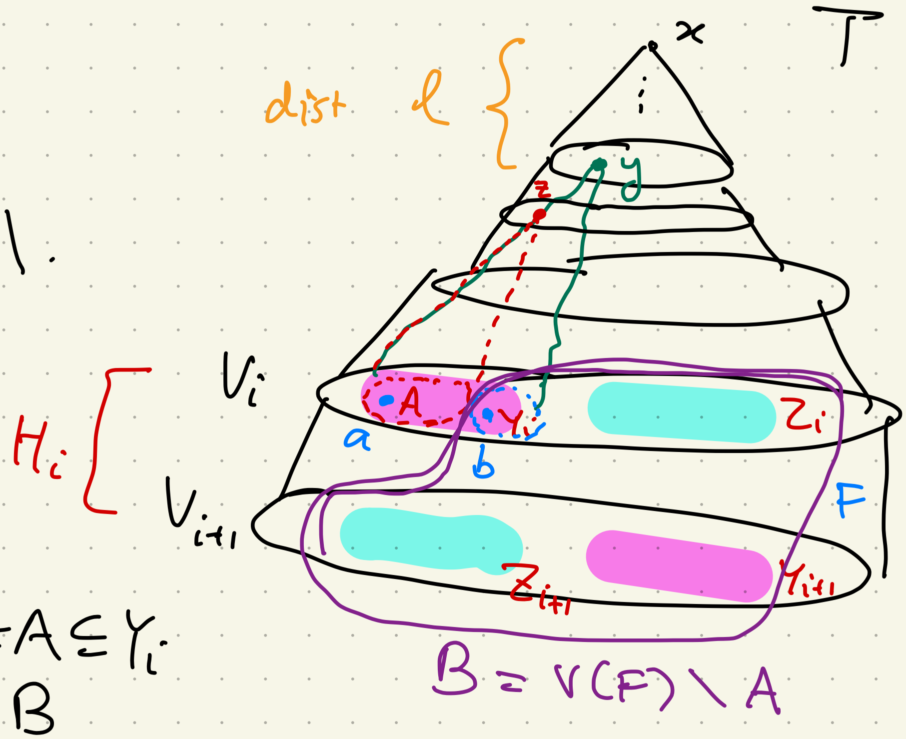
$\Rightarrow$ (A, B) is not a bipartition of F.

Lem $\Rightarrow$ $\exists$ (A, B)-path of any length $< |F|$

Let ℓ be the dist. between x and y

So $\ell < i$ and

$2k - 2(i-\ell) < 2k \leq |F|$.



- Lem $\Rightarrow$ $\exists$ a,b-path P of length

  $2k-2(i-\ell)$ and $a \in A \subseteq Y_i$
  $b \in B$

- P even length $\Rightarrow$ $b \in Y_i$
- and (Y, Z) is a bipartion

y,a-path $\cup$ y,b-path $\cup$ $P \Rightarrow C_{2k}$.