Lecture 13

- Extensions

<u>Def</u>: $ex(n, \Gamma, H)$ = max # copies of Γ in an n-vx H-free graph

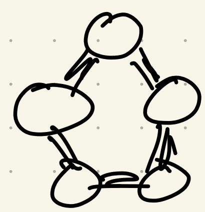
$ex(n, H) = ex(n, K_2, H)$

- Erdős-Zykov solved the clique case:

$\forall s<t$ $\quad ex(n, K_s, K_t)$ is maximised by $T_{t-1}(n)$

- Alon-Shikhelman 16
- Gishboliner-Shapira (cycles)
- Grzesik, Hatami-Hladky-Kral-Norine-Razborov

$$ex(n, C_5, K_3) = \left(\frac{n}{5}\right)^5$$



blowup of C_5

<u>Ex</u>: The problem is degenerate, i.e.

$$ex(n, \Gamma, H) = o(n^{|\Gamma|}), \quad \text{iff} \quad \exists\, H \xrightarrow{\text{hom}} \Gamma.$$

<u>Ex</u>: When it is degenerate, in fact

$$ex(n, \Gamma, H) = O(n^{|\Gamma| - c_H}).$$

<u>Q</u>: Are there any characterisation for when

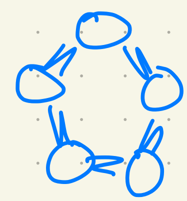
$$ex(n, \Gamma, H) = o(n^{|\Gamma| - k}) \text{ for some } k \in \mathbb{N}\,?$$

Two conj. of Erdős. ($250)

(Sparse halves)

<u>Conj</u> $\forall$ n-vx Δ-free graph G contains a set of $n/2$ vxs inducing $\leq n^2/50$ edges

<u>Rmk</u> Best possible if true



<u>Conj</u> $\forall$ n-vx Δ-free graph can be made bip. by removing $\leq n^2/25$ edges

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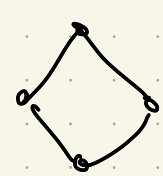
Part 2 Bipartite Turán

We have seen in ESS that $ex(n, H) = o(n^2)$ when H is bip. We shall see that in fact we have $O(n^{2-c_H})$.

- <u>Ref</u>: Füredi–Simonovits Survey : arXiv: 1306.5167.

Earliest result, Erdős, C_4 case

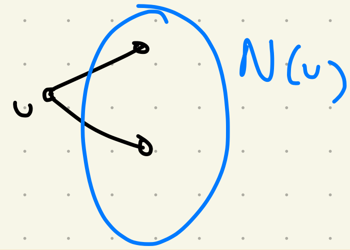


<u>Thm</u> $ex(n, C_4) \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right)$

$$= \left(\frac{1}{2}+o(1)\right) n^{3/2}$$

Pf: Let G be an n-vx C_4-free graph.

Idea: Double count 'cherries' $K_{1,2}$



- 1) Pick middle vx v

  2) two of its neighbrs.

  Take $f(x) = \binom{x}{2}$

Jensen's ineq.

f convex

$\forall$ r.v. X

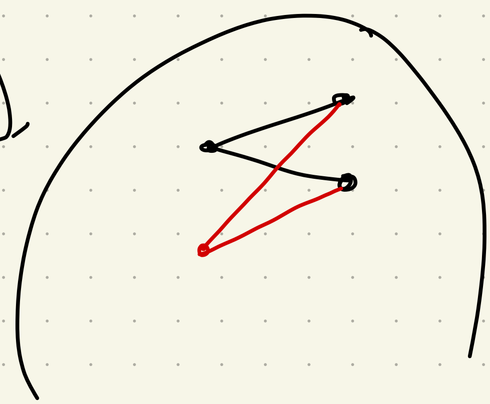
$\Rightarrow \mathbb{E}(f(X)) \geq f(\mathbb{E}X)$

$$\Rightarrow \# = \sum_{v \in V(G)} \binom{d(v)}{2}$$

$$\overset{\text{Jensen}}{\geq} n \cdot \binom{\underbrace{\frac{1}{n}\sum d(v)}_{\text{ave. deg}}}{2}$$

$$= n \binom{\frac{1}{n} 2e(G)}{2} = \frac{2\,e(G)^2}{n} - e(G)$$

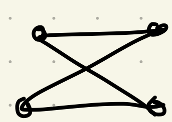
G



- On the other hand, $\# \leq \binom{n}{2}$,

for o.w. we get two $K_{1,2}$'s with diff. middle vxs sharing the same pair of end vxs, yielding a copy of C_4. ↯

Note that $C_4 = K_{2,2}$.



By double counting stars instead of cherries we get the following.

Ex $\forall s \le t$: $ex(n, K_{s,t}) \le t \cdot n^{2-1/s}$

A careful calculation using this approach gives

Thm (Kövari-Sós-Turán) $\forall s \le t$

$$\Rightarrow \quad ex(n, K_{s,t}) \le \left(\tfrac{1}{2} + o(1)\right)(t-1)^{1/s}\, n^{2-1/s}$$

As every bip. H is a subgraph of some $K_{s,t}$

$$\Rightarrow \quad ex(n, H) = O(n^{2-c_H}).$$

• It is a major open problem to determine the order of magnitude of $ex(n, K_{s,t})$

Matching lower bound constructions $\Omega(n^{2-1/s})$

are only known when • $s = 2, 3$

Norm graph - Kollár, Rónyai, Szabó - Alon, Rónyai, Szabó $\rightarrow$ • $t \geq (s-1)! + 1$

Random alg. constr. Bukh $\rightarrow$ • $t \geq$ exponential (s).

$K_{3,3}$-free constr.

Open Improve $\Omega(n^{5/3}) \le ex(n, K_{4,4}) \le O(n^{7/4})$

- C_4-free graphs and Sidon sets.

<u>Def</u>: A set $S = \{a_1, \dots, a_k\} \subseteq \mathbb{N}$ is a **Sidon set** if all pairwise sums $a_i + a_j$, $i \le j$, are distinct, i.e.

$a+b=c+d$ has only trivial sol^{n} $\{a,b\}=\{c,d\}$ in S.

<u>Thm</u>(♡) The largest Sidon set in $[n]$ has size

$$(1+o(1))\sqrt{n}.$$

<u>Ex</u> 1) Prove the weaker upp. bd that every Sidon set in $[n]$ has size $\le 2\sqrt{n}$.

2) Use Thm(♡) to construct an n-vx C_4-free graph w./ $\Omega(n^{3/2})$ edges.

This exercise shows that $ex(n, C_4) = \Theta(n^{3/2})$

<u>Thm</u> (Erdős-Rényi-Sós) $ex(n, C_4) \geq \left(\frac{1}{2} - o(1)\right) n^{3/2}$.

<u>Pf</u>: • For large enough n, it is known that $\exists$ a prime p between $(1-o(1))\sqrt{n+1}$ and $\sqrt{n+1}$.

Consider the following graph G:

- $V(G) = \mathbb{F}_p^2 \setminus \{(0,0)\}$ and

- $E(G) = \{ (a,b)(x,y) : ax + by = 1 \}$

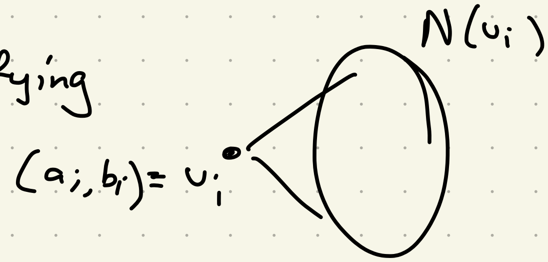
Delete loops.

Take two distinct vxs $\underline{v_i} = (a_i, b_i)$, $i \in [2]$,

$N(v_i)$ consists of vxs (x,y) satisfying

$a_i x + b_i y = 1$, which is

a line in $\mathbb{F}_p^2$.



As $v_1 \neq v_2$, $N(v_1)$ and $N(v_2)$ are two distinct lines

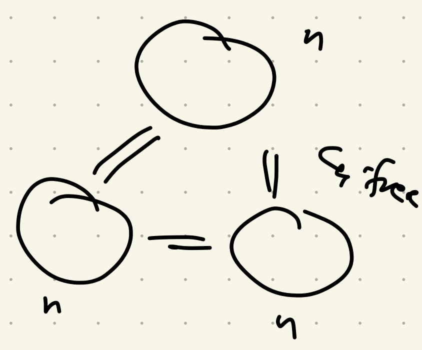
intersecting at $\leq$ one pt.

$\Rightarrow$ v_1, v_2 have ≤ 1 common neighbr. $\Rightarrow$ C_4-free.

- every vx has deg p or $p-1$ (if $\exists$ loop)

$e(G) \geq \frac{1}{2}(p-1)(p^2-1) = \left(\frac{1}{2} - o(1)\right) n^{3/2}$ ☺

A related problem asked by Fischer - Matousek.



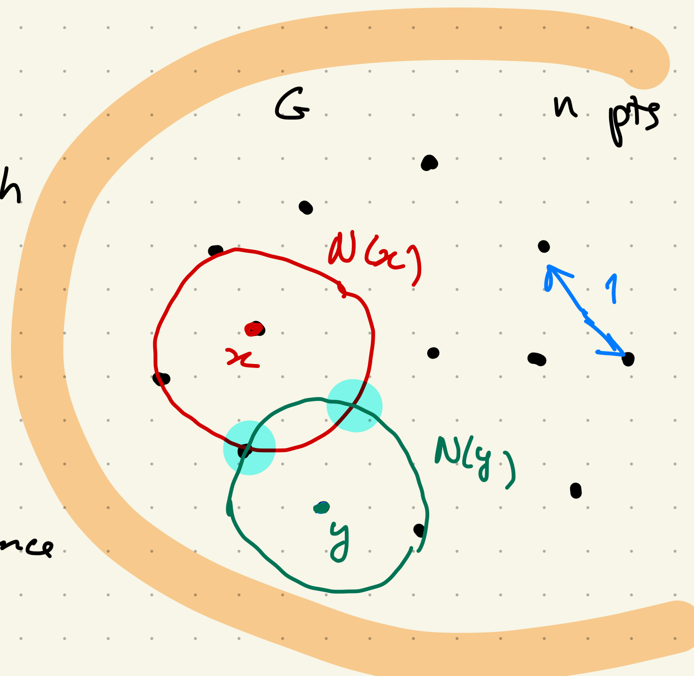
Open problem Let G be an $n \times n \times n$-vx 3-partite graph. If between any two partite sets, the induced bip. graph is C_4-free, then how many Δs can G have?

- $n^{5/3} \underset{\text{J}}{\leq} \# \underset{\text{C-S}}{\leq} n^{7/4}$

Coulter - Matthews - Timmons

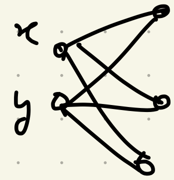
A quick application of K-S-T to Erdős unit distance problem.

<u>Prop</u> A set of n pts in $\mathbb{R}^2$ determines $\leq O(n^{3/2})$ unit distances



<u>Pf</u>: Build an auxiliary graph G w./ vx set being these n pts, where two pts are adj. iff they are of distance 1 apart.

Note that G is $K_{2,3}$-free



as $N(v)$ corresp. a unit circle centered at v and two circles in the plane intersect at $\leq$ two pts.

- $KST \Rightarrow e(G) = O(n^{3/2})$

• <u>Rmk</u> best known upp. bd is $O(n^{4/3})$ due to Szemerédi - Trotter. Improving the $4/3$ exponents would be considered a breakthrough.

• Brown's constr: $ex(n, K_{3,3}) = \Omega(n^{5/3})$

$V = \mathbb{F}_p^3$, $\underset{\overset{\shortparallel}{v}}{(a,b,c)} \sim \underset{v'}{(a',b',c')}$

$\Longleftrightarrow \|v-v'\|^2 = (a-a')^2 + (b-b')^2 + (c-c')^2 = 1$

• $K_{3,3}$-free, geometric intuition: $N(v)$ corresp. a sphere around v and three spheres have ≤ 2 common pts.

• Even cycles

- Bondy - Simonovits $ex(n, C_{2k}) = O(n^{1+1/k})$

Only matching lower bds known: $k = 2, 3, 5$ coming from finite geometry.

Lazebnik - Ustimenko - Woldar.

<u>Open</u> Improve $\Omega(n^{6/5}) \leq ex(n, C_8) \leq O(n^{5/4})$

We will present a short pf of Pikhurko using BFS.

Thm (Pikhurko) For large n,

$$ex(n, C_{2k}) \leq 4k \cdot n^{1+1/k}.$$

Rmk: • Split opinions on whether $n^{1+1/k}$ is correct

• The leading constant has been brought down to

[He] $O(\sqrt{k \log k} \cdot n^{1+1/k})$

Open Is $ex(n, C_{2k}) = o(\sqrt{k} \cdot n^{1+1/k})$?