Lecture 12

Recall For a graph H, its homomorphism threshold is

$$\delta_{hom}(H) = \inf \{ \alpha \in [0,1] : \exists \overset{\text{H-free graph}}{\Gamma = \Gamma(H,\alpha)} \text{ s.t. } \forall \text{ n-vx H-free } G \text{ w./ } \delta(G) \geq \alpha n \text{ satisfies } G \xrightarrow{hom} \Gamma \}$$

- Since $G \xrightarrow{hom} \Gamma \Rightarrow \chi(G) \leq |\Gamma| = O_{H,\alpha}(1)$

$\Rightarrow \forall H, \quad \delta_{hom}(H) \geq \delta_{\chi}(H)$

So $\delta_{hom}(K_3) \geq \delta_{\chi}(H) = 1/3$

Today : $\delta_{hom}(K_3) \leq 1/3$

Thm (Oberkampf - Schacht 2020)

$\forall$ n-vx Δ-free graph G w./ $\delta(G) \geq (\frac{1}{3} + \varepsilon) n$,

$\exists$ Δ-free graph Γ w./ $|\Gamma| \leq 2^{2^{\mathrm{poly}(1/\varepsilon)}}$ s.t. $G \xrightarrow{hom} \Gamma$.

The pf is probabilistic (showing a stronger statement that G is in fact a blowup of such Γ. if G is maximal Δ-free)

Idea Partition $V(G)$ according to a random set (of constant size) and show that this ptt. corresponds to a blowup of some Δ-free Γ.

- Notation: for $U \subseteq V(G)$, $N^*(U)$ = common neighbrhd $= \bigcap_{u \in U} N(u)$.

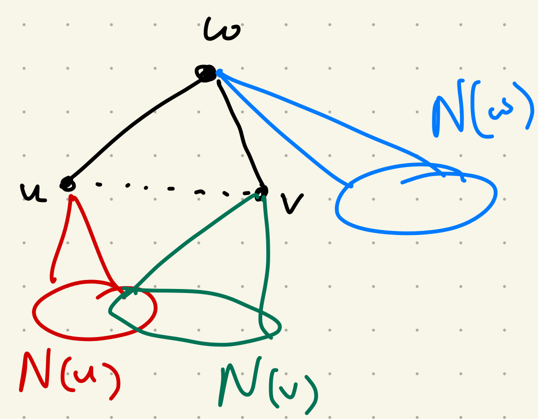
By pigeonhole.

Obs $|N^*(U)| \geq \delta(G)\cdot|U| - (|U|-1)n$. Exercise if not clear.

Prop G n-vx maximal Δ-free, $\cdot\ \delta(G) \geq (\frac{1}{3}+\varepsilon)n$ $\implies$ $\forall$ nonadjacent u,v $|N(u) \cap N(v)| \geq 3\varepsilon n$

Pf: • maximality $\Rightarrow$ diam ≤ 2

$\exists\ w \in N(u) \cap N(v)$



- Δ-free $\Rightarrow$ $N(w) \cap N(u) = \phi$, $N(w) \cap N(v) = \phi$ $\Rightarrow |N(u) \cup N(v)| \leq n - d(w)$

$$\Rightarrow |N(u) \cap N(v)| = d(u) + d(v) - |N(u) \cup N(v)| \quad \leq (\tfrac{2}{3} - \varepsilon)n$$

$$\geq 2\left(\tfrac{1}{3}+\varepsilon\right)n - \left(\tfrac{2}{3}-\varepsilon\right)n = 3\varepsilon n.$$

Pf ($\delta_{hom}(K_3) \le 1/3$)

Set $m = \frac{8 \ln \frac{8}{\varepsilon}}{\varepsilon^2}$ $(= O_\varepsilon(1))$, $T = 2^m$, $L = 2^T + T$

We may assume that G is maximal Δ-free, we will show that G is a blowup of some Δ-free Γ w/ $|\Gamma| \le L$.

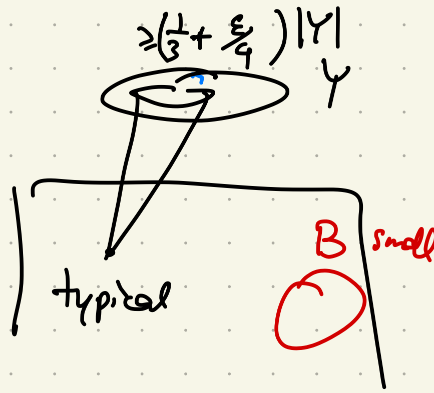
Claim $\exists$ a set Y w/ $m \ge |Y| \ge (1 - \frac{\varepsilon}{4}) m$ s.t.

(i) (few bad vxs wrt Y)

$B = B(Y) = \{ v \in V(G) : d(v, Y) < (\frac{1}{3} + \frac{\varepsilon}{4}) |Y| \}$

has size $|B| \le \varepsilon n / 4$

(ii) $\delta(G[Y]) \ge (\frac{1}{3} + \frac{\varepsilon}{4}) |Y|$.



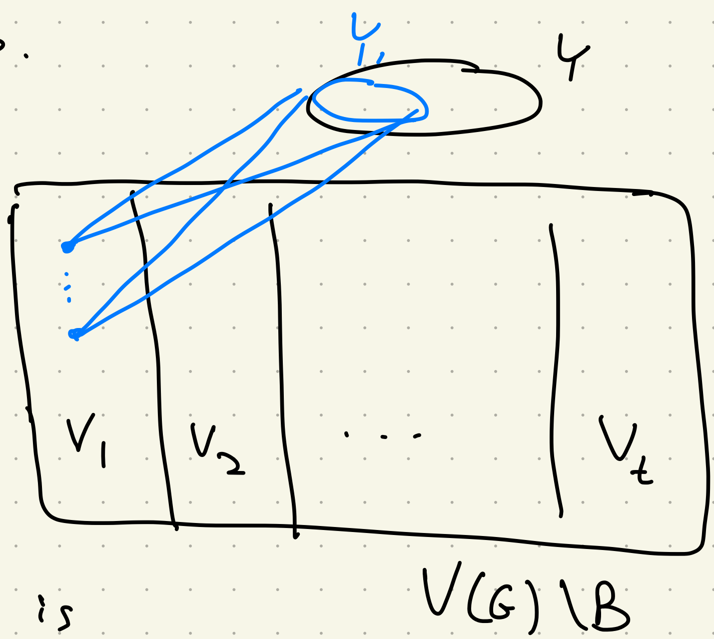
Partitioning typical vxs $V(G) \setminus B = V_1 \cup \cdots \cup V_t$

according to the neighborhd in Y, i.e.

$\forall i \in [t]$, $\forall v \in V_i$,

$N(v, Y) = Y_i \subseteq Y$

so $t \le 2^{|Y|} \le 2^m = T$



We shall see that

$(V_1, \ldots, V_t)$ is a blowup, that is

- V_i indep $\forall i \in [t]$
- V_i, V_j either completely joined $\forall i \ne j \in [t]$

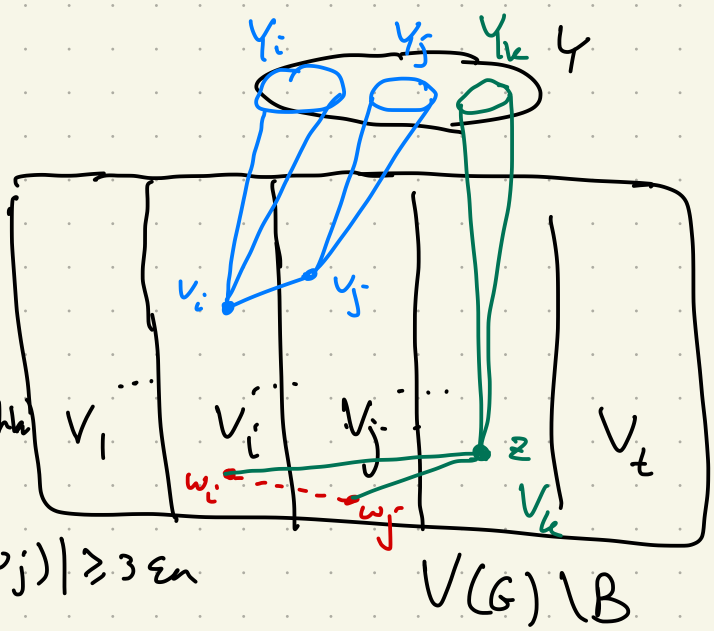
or no edge between

- By defn of B, $|Y_i| \geq \left(\frac{1}{3} + \frac{\varepsilon}{4}\right)|Y| > 0$, $\forall i \in [t]$.

  which together w./ Δ-freeness $\Rightarrow$ V_i indep.

- Supp. $\exists\ v_i \sim v_j$ $\quad v_i, w_i \in V_i$

  $w_i \not\sim w_j$ $\quad v_j, w_j \in V_j$



- $v_i \sim v_j \Rightarrow Y_i \cap Y_j = \phi$

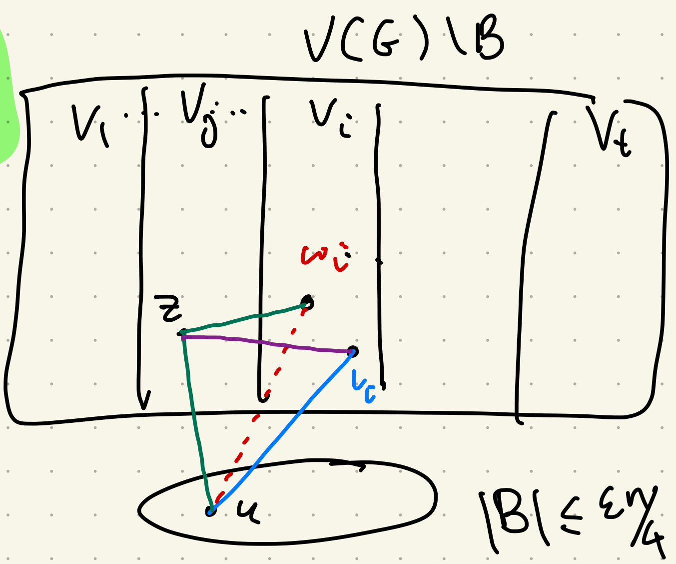
- $w_i \not\sim w_j \xRightarrow{\text{Prop}}$ w_i, w_j have a common neighb in $V(G) \setminus B$ as $\begin{cases} |N(w_i) \cap N(w_j)| \geq 3\varepsilon n \\ |B| \leq \varepsilon n/4 \end{cases}$

Say $z \in V_k$, $z \sim w_i, w_j \Rightarrow Y_k$ disjoint from both Y_i, Y_j as G is Δ-free.

But $|Y_i|, |Y_j|, |Y_k|$

$\geq \left(\frac{1}{3} + \frac{\varepsilon}{4}\right)|Y|$ ↯.

Dealing w./ bad vxs in B



Let us first show that

$\forall\ u \in B$, $\forall i \in [t]$,

either $N(u, V_i) = V_i$

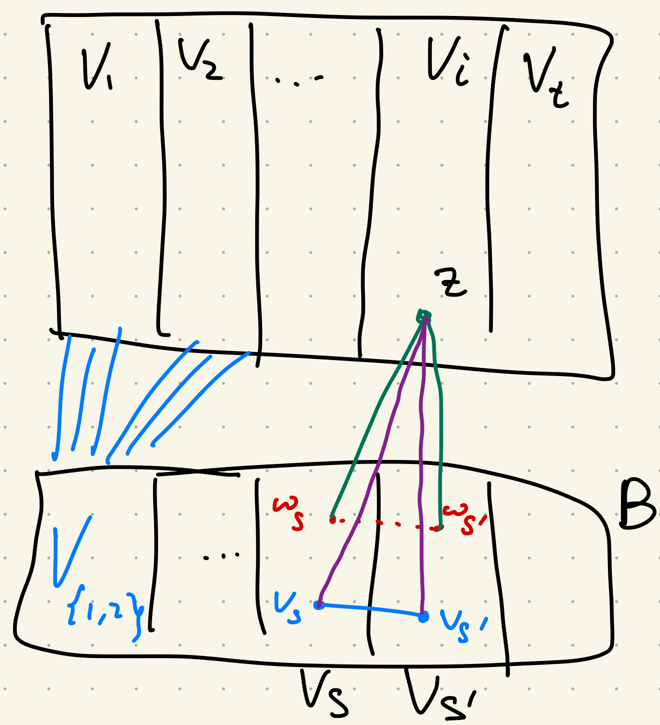
or $N(u, V_i) = \phi$

Supp. $\exists\, v_i, w_i \in V_i$, s.t. $u \sim v_i$, $u \not\sim w_i$

Again, as $u \not\sim w_i$, by prop. u & w_i have a common neighbr z in $V(G)\setminus B$, say $z \in V_j$

Now $z \sim w_i \Rightarrow V_i\, V_j$ complete $\Rightarrow z\, v_i\, u \cong \Delta$ $\lightning$.

• Thus, we can partition B as follows



$$B = \bigcup_{S \subseteq [t]} V_S$$

$\Rightarrow$ a ptt. of $V(G)$ w/

$\leq t + 2^t \leq T + 2^T = L$ parts

We are left to show that

$\bigcup_{S \subseteq [t]} V_S$ is a blowup, i.e.

• each V_S indep set $\Leftarrow$

v_S in V_S each has $\geq \delta(G) - |B| \geq (\frac{1}{3} + \frac{3\varepsilon}{4})n$ neighbrs in $V_1 \cup \cdots \cup V_t$ and they have the same neighbhd in $V_1 \cup \cdots \cup V_t$

With Δ-freeness $\Rightarrow$ V_S indep.

• $V_S, V_{S'}$ $\begin{cases} \text{completely joined} \\ \text{OR} \\ \text{no edge in between} \end{cases}$ $\forall S \neq S' \subseteq [t]$

Suppose $w_S \not\sim w_{S'}$, $v_S \sim v_{S'}$

Prop $\Rightarrow \exists\, z \in V_i$ (some) $z \sim w_S, w_{S'}$

$\Rightarrow v_S, v_{S'} \sim z \Rightarrow \Delta$ $\lightning$ ☺