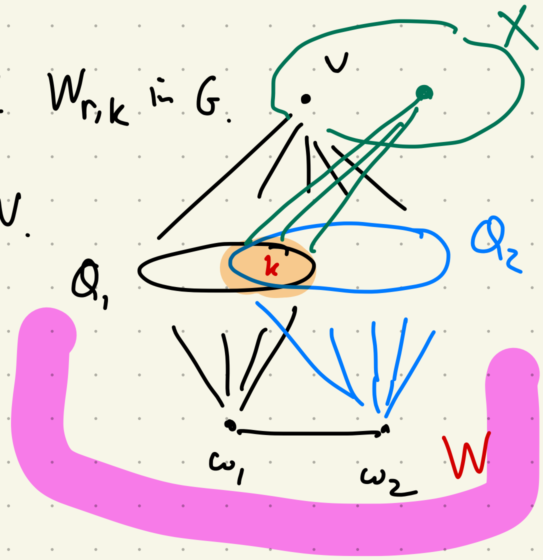
Lecture 9

<u>Prop 1</u> G maximal K_r-free, not $(r-1)$-partite $\Rightarrow \exists\, W_{r,k} \subseteq G$.

<u>Pf</u> ($\delta(G) > \frac{3r-7}{3r-4} n$, maximal K_r-free $\Rightarrow$ $(r-1)$-partite)



Supp. not, by Prop 1, there is a copy of $W_{r,k}$ in G.

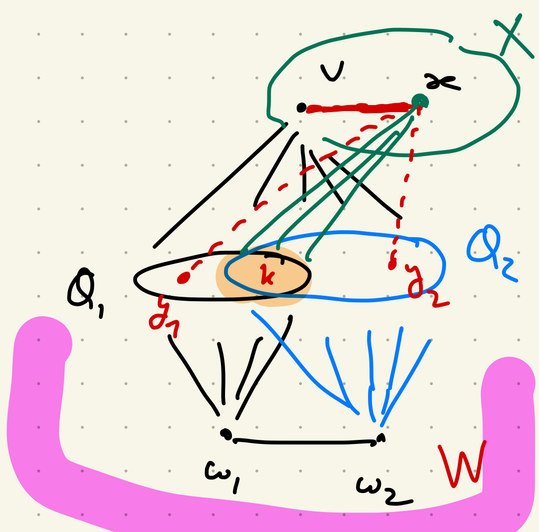
Take one w/ maximum k, say on vertex set W.

Let $p := |W| = 1 + 2 + 2(r-2) - k$

$= 2r - k - 1$.

- As W contains a K_{r-1}, every vx has deg $\le p-1$ in W.

- Let X be the common neighbrhd of $Q_1 \cap Q_2$



<u>Claim</u> $\forall\, x \in X$, $d(v, W) \le p - 3$

<u>Pf</u>: • K_r-free $\Rightarrow$ x has a non-neighbor in $Q_i \cup \{w_i\}$, for each $i \in [2]$

- Suppose $x \sim v$, then K_r-free $\Rightarrow$

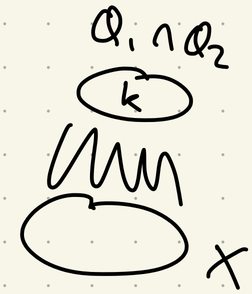
$\forall i \in [2]$, $\exists\, y_i \in Q_i \setminus Q_{3-i}$ s.t. $x \not\sim y_i$

Now if $x \sim$ both w_1 and w_2,

then $W - y_1 - y_2 + x$ induces a copy of $W_{r,k+1}$, ↯ max. k.

• By the claim,

$$(1) \cdots \quad \delta(G)\cdot p \le \sum_{w\in W} d(w) \le \overbrace{(p-3)|X| + (p-1)(n-|X|)}^{\text{sum deg of all } u\text{s to } W}$$



By pigeonhole,

• $|X| \geq k\cdot\delta(G) - (k-1)n \quad \cdots \quad (2)$

$$(1) \,\&\, (2) \Rightarrow \quad \delta(G) \le \frac{2r+k-4}{2r+k-1}\cdot n$$

As $k \le r-3$, $\Rightarrow \quad \delta(G) \le \dfrac{3r-7}{3r-4}\cdot n \quad$ ↯.

Recall Δ-case: n-vertex, Δ-free, $\delta(G) > 2n/5 \Rightarrow$ bip. $\quad \chi(G)\le 2$

Q. How much we can lower the mindeg if we only ask for bdd chr. number? best: $1/3$

dichotomy $\forall \varepsilon, \exists C = C(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ s.t. TFH $\forall n \ge n_0$,

- Thomassen: $\forall n$-vertex, Δ-free, $\delta(G) \ge (\frac{1}{3}+\varepsilon)n \Rightarrow \chi(G) \le C$
- Hajnal: $\exists$ n-vertex, Δ-free, $\delta(G) = (\frac{1}{3} - o(1))n$ and arbitrarily large chr. number.

Related work

- Brandt - Thomassé: $\forall$ n-vx, Δ-free, $\delta(G) > \frac{n}{3} \Rightarrow \chi(G) \le 4$.
- Goddard-Lyle: $\forall n$-vx, K_r-free, $\delta(G) > \frac{2r-5}{2r-3}n \Rightarrow \chi(G) \le r+1$

Def: (Chr. threshold) The chromatic threshold of a graph H is

$$\delta_\chi(H) := \inf\left\{ \alpha \in [0,1] : \exists\, C(H,\alpha) \text{ s.t. } \forall \text{ n-vx H-free graph } G \text{ with } \delta(G) \geq \alpha n \Rightarrow \chi(G) \leq C \right\}$$

That is, $\forall\, \alpha < \delta_\chi(H)$, the chr. # of n-vx H-free graph w/ mindeg αn could be arbitrarily large; while when $\alpha > \delta_\chi(H)$, then the chr. # is necessarily bounded.

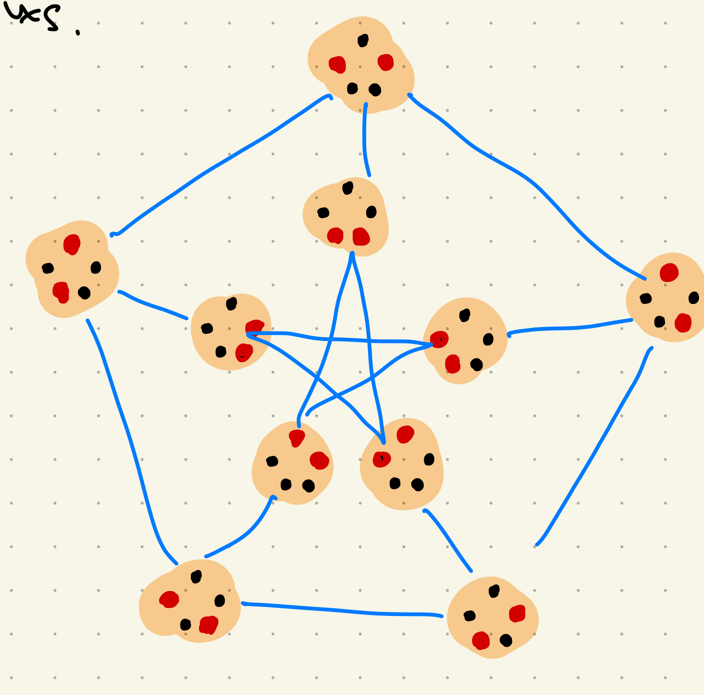
$\delta_\chi(K_3) = 1/3$, $\qquad \delta_\chi(K_r) = \dfrac{2r-5}{2r-3}$.

- Construction of dense Δ-free graphs w/ large chr. #.

Goal: n-vx Δ-free $\delta(G) = (1/3 - o(1))n$, large chr. #.

Def. (Kneser graph) The Kneser graph $KG(n,k)$ has vertex set $V = \binom{[n]}{k}$ and two vxs S, T are adj if $S \cap T = \phi$.

- $KG(n,1) = K_n$ compl. on n vxs.
- $KG(5,2) =$ Petersen gr.



$KG(n,k)$ has $\binom{n}{k}$ vxs

$\binom{n-k}{k}$-reg

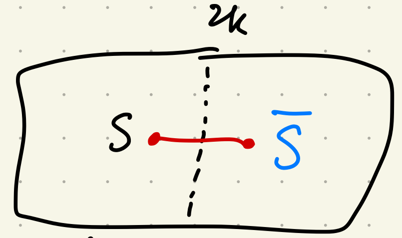
Thm (Lovász) $\chi(KG(n,k)) = n-2k+2$.

$\forall$ vertex $S \in \binom{[n]}{k}$, if $\min S \leq n-2k$, then color S w/ color $\min S$.

$(\leq)$ $\forall\, i \in [n-2k]$, color all k-sets containing i w/ color i.

The remaining uncolored k-sets all lie in $\{n-2k+1, \ldots, n\}$, a set of size $2k$.

The Kneser gr induced on this part is a perfect matching, which is 2-colorable.



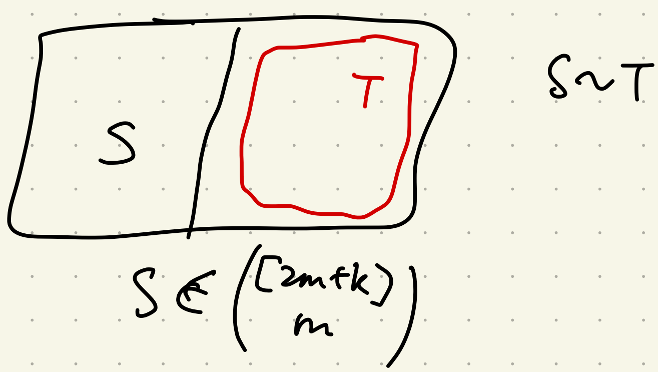
- As a consequence, we obtain graphs w/ large chr. # and large odd girth (i.e. shortest odd cycle is long)

Take $1 \ll k \ll m$, consider $KG(2m+k, m) = G$

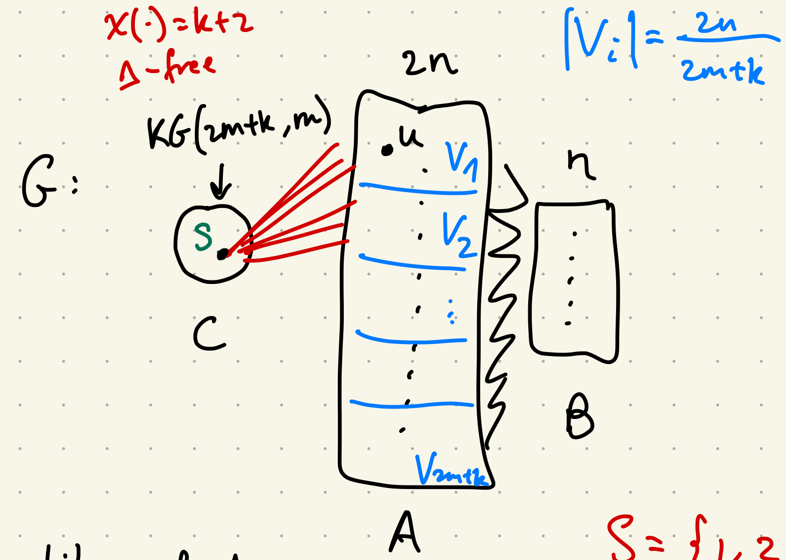
- Lovász $\Rightarrow$ $\chi(G) = 2m+k-2m+2 = k+2$

Not hard to see G has no short odd cycle.

In particular, G is Δ-free.



Hajnal's construction



$S = \{1, 2\}$

- Take $1 \ll k \ll m \ll n$
  - $C \cong KG(2m+k, m)$
  - $[A, B]$ induces $K_{2n,n}$
  - Take a $(2m+k)$-equipartition of A:

    $A = V_1 \cup \cdots \cup V_{2m+k}$

  - A vertex $S \in \binom{[2m+k]}{m} \sim V_i$ if $i \in S$
  - $\delta(G) \geq \frac{m}{2m+k} \cdot 2n = \left(\frac{1}{3} - o(1)\right) |G|$.
  - To show Δ-freeness, it suffices to show that $\forall\, u \in A$ $N(u, C)$ is an indep. set. Wlog $u \in V_1$

    $\Rightarrow$ $u \sim$ all m-sets containing 1, which form an indep set as desired.

- For generic H,
  - if H bip., ESS $\Rightarrow$ $\delta_\chi(H) = 0$
  - Thomassen: $\delta_\chi(C_{2k+1}) = 0$, $\forall\, k \geq 2$.

    That is, $\forall$ n-vx, C_{2k+1}-free graph G w/ $\delta(G) = \Omega(n)$

    $\chi(G) = O(1)$.

- Allen - Böttcher - Griffiths - Kohayakawa - Morris determines the chr. threshold $\delta_\chi(H)$ for all H.