Lecture 8

Recent developments on Stability results.

- L - Pikhurko - Sharifzadeh - Staden

  Axiomatized pf, giving suff. cond. guaranteeing perfect stability, application on inducibility problem.
- Balogh - Clemen - Lavrov - Lidický - Pfender

  Making K_{r+1}-free graphs r-partite
- Exact Stability for Turán's thm

  Korándi - Roberts - Scott.

§ Stability method

Tackle an extremal problem

Stab. method
- Step 1 : Obtain asymptotic result;
- Step 2 : Obtain Stability result;
- Step 3 : Stability $\Rightarrow$ exact.

As an illustration, we prove the following special case of Thm ... (color-critical)

Thm For large n, $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$.

- Step 1 : ESS
- Step 2 : E-Sim stability.

Lem 1 Let $\alpha > 0$, there exists n_0 s.t. TFH for $n \geq n_0$.

Let G be an n-vertex C_{2k+1}-free graph.

If $e(G) \geq \frac{n^2}{4} - \alpha n^2$,

$\Longrightarrow$ then G can be made bipartite by removing $\leq 5\alpha n^2$ edges.

Pf (Thm) Let G be an n-vx extremal C_{2k+1}-free graph. Lower bound $e(G) \geq e(T_{n,2}) = \lfloor n^2/4 \rfloor$.

Upp bd : • first reduce it to graph w/ high min deg.

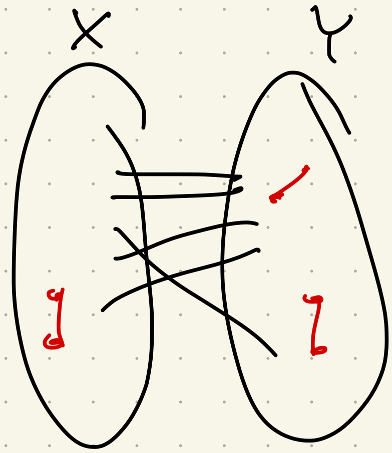
Claim $\exists\, G' \subseteq G$ w/ $|G'| \geq n/2$ and

$$\delta(G') \geq \left(\tfrac{1}{2} - \varepsilon\right)|G'|.$$

Pf : Repeatedly remove low deg vxs.

Exer. $\blacksquare$

• Let $V(G') = X \cup Y$ be a max-cut for G'.



By Stability $\Rightarrow e_{G'}(X) + e_{G'}(Y) \leq 5\varepsilon (n')^2/2$

$\Rightarrow |X| = |Y| = \frac{n'}{2} \pm 3\sqrt{\varepsilon} n'$.

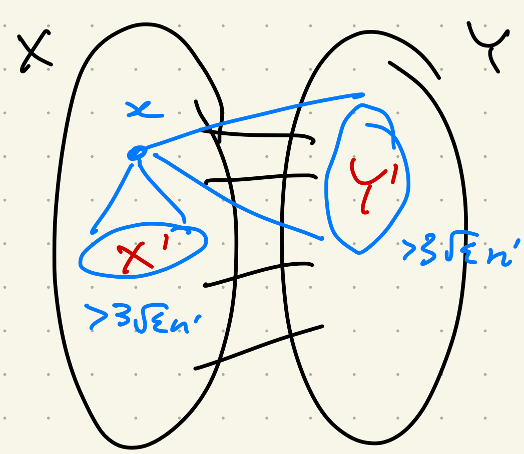
Otherwise,

$$e(G') \leq |X||Y| + e_{G'}(X) + e_{G'}(Y)$$
$$\leq \frac{(n')^2}{4} - 9\varepsilon (n')^2 + 5\varepsilon (n')^2/2$$
$$\leq \frac{(n')^2}{4} - 5\varepsilon (n')^2 \quad \lightning$$

• Next, we show $\Delta(G'[X]), \Delta(G'[Y]) \leq 3\sqrt{\varepsilon} n'$

Supp. $\exists x \in X$ w./ $d(x, X) > 3\sqrt{\varepsilon} n'$.

Max-cut $\Rightarrow d(x, Y) \geq d(x, X) > 3\sqrt{\varepsilon} n'$.



$G'[N(x, X), N(x, Y)]$

is P_{2k}-free as G' is C_{2k+1}-free

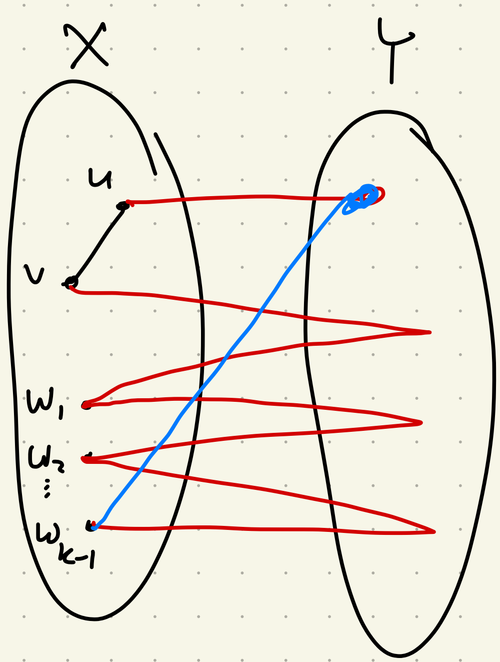
Erdős-Gallai $\Rightarrow$ $e(X', Y') = O_k(n')$

$$\Rightarrow e(G') \leq |X||Y| - \left(|X'||Y'| - O_k(n) \right) + 5\varepsilon(n')^2/2$$

$$\leq \frac{(n')^2}{4} - 2\varepsilon(n')^2 \quad ↯$$

- Supp $\exists$ edge $uv \in X$ (same pf for Y)

Take $w_1, \dots, w_{k-1} \in X$.



- $\Delta(G'[X]) \leq 3\sqrt{\varepsilon} n'$
- $|X| = |Y| = n'/2 \pm 3\sqrt{\varepsilon} n'$
- $\delta(G') \geq (1/2 - \varepsilon) n'$

$\Rightarrow \forall x \in X$

$$d(x, Y) \geq \delta(G') - \Delta(G'[X])$$

$$\geq \left(\frac{1}{2} - 4\sqrt{\varepsilon}\right) n'$$

$$\geq (1 - 20\sqrt{\varepsilon}) |Y|$$

$\Rightarrow$ $u, v, w_1, \dots, w_{k-1}$ have $\geq (1 - 20(k+1)\sqrt{\varepsilon})|Y|$

common neighbors in Y

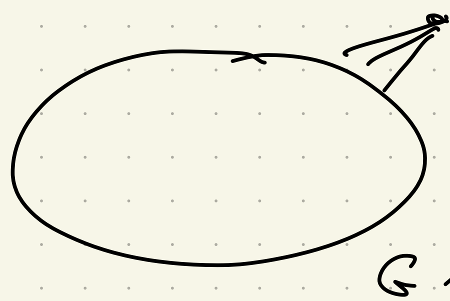
$\Rightarrow C_{2k+1}$. ↯

Thus, G' is bipartite.

$$e(G) \le e(G') + \sum_{n'+1}^{n} \left(\tfrac{1}{2}-\varepsilon\right) i$$

$$< \frac{n^2}{4}$$

if $n' < n$.



$\Rightarrow G' = G$ bip. $\Rightarrow G = T_{n,2}$. ▨

§ K_r-free graphs w/ high min deg.

Thm (Andrasfai-Erdős-Sós)

$\forall$ n-vx, K_r-free graph G

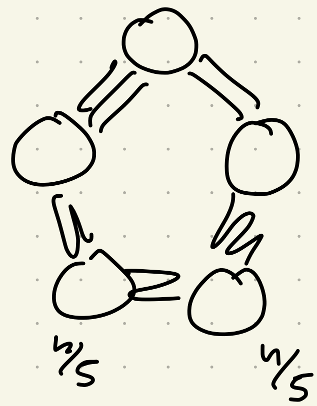
if $\delta(G) > \frac{3r-7}{3r-4} n \Rightarrow G$ is $(r-1)$-partite

For the Δ-case

<u>Thm</u> $\forall$ n-vx Δ-free $\delta(G) > \frac{2}{5}n$

$\Longrightarrow$ G is bipartite.

- Tight : consider



$5/n$

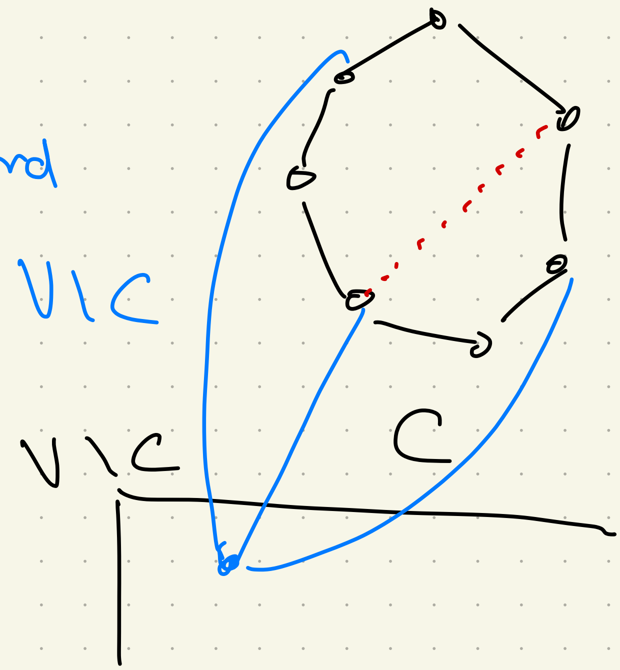
<u>Pf</u>: • Supp. G is not bipartite

$\Longrightarrow$ G has an odd cycle.

Take an odd cycle C w/ minimum length, say w/ vx set $(x_1, \dots, x_t)$

Note that minimality of C $\Longrightarrow$ i) C has no chord

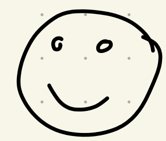
ii) every vx $v \in V \setminus C$ has ≤ 2 neighbors in C



$t \geq 5$ odd

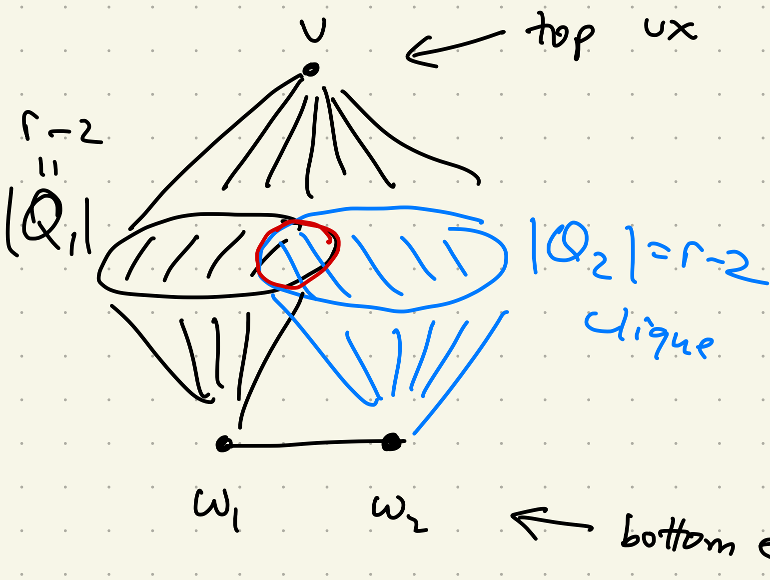
- $$\sum_{x\in C} d(x) > |C|\cdot\frac{2n}{5} = 2n$$

- $$\sum_{x\in C} d(x) = 2e_G(V(C)) + e_G(V(C), V\setminus C)$$
$$\leq 2t + 2(n-t) = 2n$$



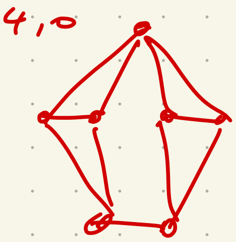
For the general case, we will present a short pf by Brandt.

Def (5-wheel-like graph $W_{r,k}$)

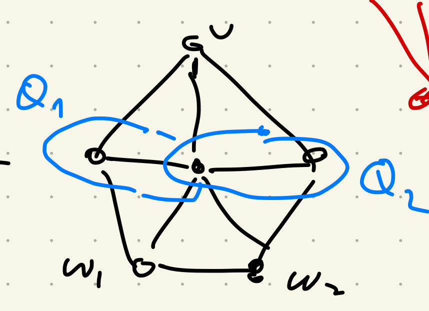


- $v \sim Q_1$ & Q_2
- Q_1, Q_2 cliques size $r-2$
- $|Q_1 \cap Q_2| = k$
  $0 \leq k \leq r-3$
- $w_1 \sim Q_1$
- $w_2 \sim Q_2$

$W_{4,0}$



Ex $W_{3,0} = C_5$, $W_{4,1} =$



We need the following obs. extending the fact that every maximal Δ-free graph is either bipartite or contains a 5-cycle.

<u>Prop</u> G maximal K_r-free.

If G is not $(r-1)$-partite

$\Rightarrow$ then G contains a 5-wheel like graph.

<u>Pf</u>: • If G is complete partite

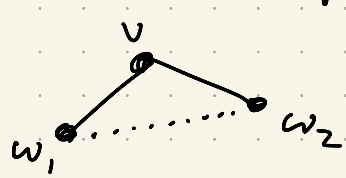
$\Longrightarrow$ # parts $\leq r-1$ as G is K_r-free

↯.

• As G is not complete partite

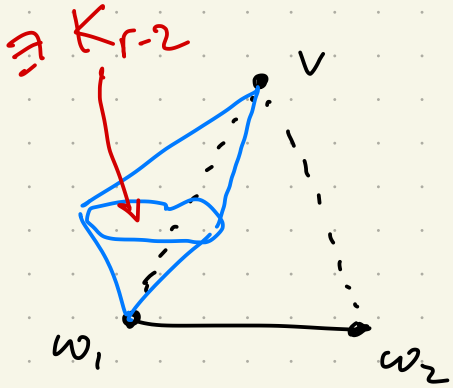
$\Rightarrow$ $\overline{G}$ is not union of cliques

$\Rightarrow$ $\overline{G}$ has an induced path P_3



$\Rightarrow$ G has

an induced $\overline{P_3}$ $\Rightarrow$



As G is maximal K_r-free.

$\forall i \in \{1,2\}$ $N(v) \cap N(w_i)$ contains a K_{r-2}

for otherwise $G + vw_i$ is still K_r-free.

$\Rightarrow$ 5-wheel-like $W_{r,k}$. ☺.