Lecture 7

Define Turán density for k-unif hyp. in an analogous way

Ex: Show that for any k-graph H, its Turán density exists.

Major open problem Determine Turán density for tetrahedron $K_4^{(3)}$, the 3-unif complete hyp. on 4 vxs.

Rmk: Although the Turán density is unknown for most of the hyp., the supersaturation pheaonmenon occurs for hyp.

Ex: Prove supersaturation for hyp.

We have seen supersaturation in graphs w/ edge density above the extremal one.

What about graphs with edge density right below the extremal density? Stability

It turns out such graphs are structurally close to the (unique) extremal graph.

Thm (Erdős-Simonovits) $\forall \alpha > 0$ and graph H with $\chi(H) = r+1$, $\exists\, n_0 = n_0(\alpha, H)$ s.t. TFH for all $n \geq n_0$.

Let G be an n-vx H-free graph.

If $e(G) \geq \left(1 - \frac{1}{r} - \alpha\right)\binom{n}{2}$

$\Rightarrow$ then G can be made r-partite by removing $\leq 5\alpha n^2$ edges

and $|E(G) \,\Delta\, E(T_{n,r})| \leq 8\sqrt{\alpha}\, n^2$.

$\uparrow$

$A \Delta B := (A \setminus B) \cup (B \setminus A)$

symm. difference

§ Perfect Stability for cliques

Thm (Füredi 15) Let G be an n-vx K_{r+1}-free graph with $e(G) = e(T_{n,r}) - t$.

Then G can be made r-partite by removing $\leq t$ edges.

(i.e. $\exists\, G' \subseteq G$, $e(G') \geq e(G) - t$, $\chi(G') \leq r$)

Furthermore, there exists a complete r-partite graph K w/ $V(K) = V(G)$ s.t. $|E(G) \,\Delta\, E(K)| \leq 3t$.

Rmk: 1) The perfect stability does not have ε, n_0, it works for all n and t.

2) It gives a linear relation between the edit distance $E(G) \triangle E(K)$ and $e(T_{n,r}) - e(G)$.

Ex (♡) Let $K = K_{n_1, \dots, n_r}$ be an n-vertex complete r-partite graph with $e(K) \geqslant e(T_{n,r}) - 2t$.

Show that $\sum (n_i - n/r)^2 \leq 4t$ and that

$$|E(K) \triangle E(T_{n,r})| \leq 2n \cdot \sqrt{t/r}.$$

The pf uses a degree majorization argument of Erdős.

Thm (Füredi 15) Let G be an n-vx K_{r+1}-free graph with $e(G) = e(T_{n,r}) - t$.
Then G can be made r-partite by removing $\leq t$ edges.

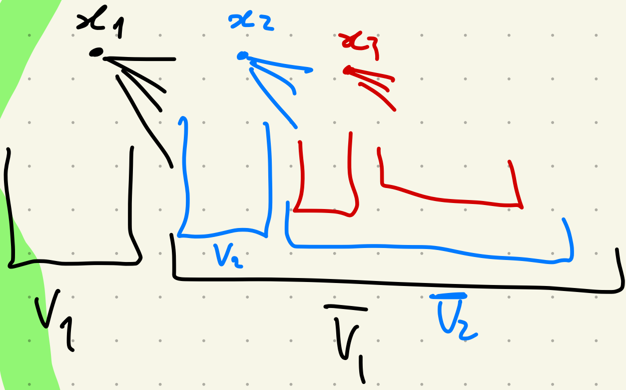
Pf It suffices to find a partition $V(G) = V_1 \cup \dots \cup V_r$ s.t. $\sum_{i=1}^{r} e_G(V_i) \leq t$

- Let $x_1 \in V(G)$ be a vertex of maximum deg.

Let $V_1 = V \setminus N(x_1)$ and $\overline{V_1} = N(x_1)$

- As $d(x) \leq d(x_1)$ $\forall x \in V_1$

$$\Rightarrow |V_1||\overline{V_1}| \geqslant \sum_{x \in V_1} d(x) = e(V_1, \overline{V_1}) + 2e_G(V_1)$$



- In general, let $\overline{V_0} = V$ and x_i be a vertex of maximum deg. in $G[\overline{V_{i-1}}]$, Let $V_i = \overline{V_{i-1}} \setminus N(x_i)$ and

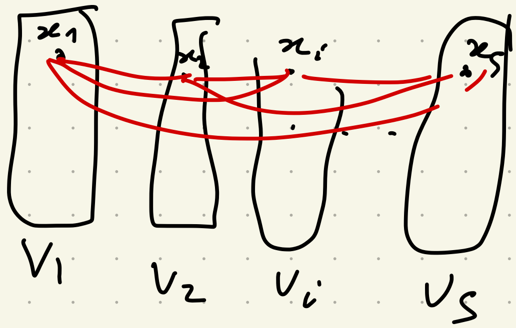
$\overline{V}_i = \overline{V}_{i-1} \cap N(x_i)$.

Then $x_i \in V_i$, $\forall x \in V_i$ $d(x, \overline{V}_{i-1}) \le |\overline{V}_i|$ by the choice of x_i

$$(*) : \Rightarrow |V_i||\overline{V}_i| \ge \sum_{x \in V_i} d(x, \overline{V}_{i-1}) = e(V_i, \overline{V}_i) + 2e_G(V_i)$$

At the end of the process, we get a partition $V = V_1 \dot\cup \cdots \dot\cup V_s$.

Note that $s \le r$ as $x_1, \dots, x_s$ form a clique, and G is K_{r+1}-free.

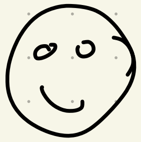


- Summing up $(*)$ $\Rightarrow$

$$e(T_{n,r}) \ge e(K_{|V_1|,\dots,|V_s|}) = \sum_{i=1}^{s} |V_i||\overline{V}_i| \ge e(G) + \underline{\overset{\#\text{ inner edges}}{\sum_{i=1}^{r} e_G(V_i)}}$$

$$\Rightarrow \quad \sum_{i=1}^{r} e_G(V_i) \le e(T_{n,r}) - e(G) = t.$$

We leave the "Furthermore" part as exercise.



Rmk: The degree majorization argument of Erdős is useful also for hypergraphs, see Mubayi 2006 [A hyp. extension of Turán thm.]

- Deriving Erdős-Simonovits Stability from Füredi Stability for cliques and removal lemma.

Lem (Ruzsa-Szemerédi removal lem)

$\forall\ \alpha > 0$ and graph H w./ $\chi(H) = t$, $\exists\ n_0 = n_0(\alpha, H)$ s.t. TFH for all $n \geq n_0$.

Let G be an n-vertex H-free graph.

Then G contains a K_t-free subgraph G' w./

$$e(G') \geq e(G) - \alpha n^2.$$

Thm (Erdős-Simonovits) $\forall\ \alpha > 0$ and graph H with $\chi(H) = r+1$, $\exists\ n_0 = n_0(\alpha, H)$ s.t. TFH for all $n \geq n_0$.

Let G be an n-vx H-free graph.

If $e(G) \geq \left(1 - \frac{1}{r} - \alpha\right)\binom{n}{2}$

$\Rightarrow$ then G can be made r-partite by removing $\leq \overset{3}{\cancel{5}}\alpha n^2$ edge

and $|E(G) \triangle E(T_{n,r})| \leq 8\sqrt{\alpha}\, n^2$.

Pf (E-Sim Stability)

- Apply removal lem on G, we get $G_1 \subseteq G$ s.t.

$$\begin{cases} \circ\ e(G_1) \geq e(G) - \alpha n^2 \geq e(T_{n,r}) - 2\alpha n^2 \\ \circ\ G_1 \text{ is } K_{r+1}\text{-free} \end{cases}$$

- By Perfect Stability for cliques $\Rightarrow$ G_1 can be made r-partite by removing $\leq 2\alpha n^2$ edges, i.e.

$\Rightarrow$ G can be made r-partite by removing $\leq 3\alpha n^2$ edges.

(i.e. $G_2 \subseteq G_1 \subseteq G$, $e(G_2) \geq e(G) - 3\alpha n^2$

$\geq e(T_{n,r}) - 4\alpha n^2$

and G_2 is r-partite

$\Longrightarrow \exists$ complete r-partite K s.t.

$$|E(G) \triangle E(K)| \leq 8\alpha n^2 .$$

Use Exer (♡) to show $|E(G) \triangle E(T_{n,r})| \leq 8\sqrt{\alpha} n^2$. ▨