Lecture 6

$k_r(G) = \#\, K_r$'s in G

$\mathcal{K}_r(G) =$ fam. of all copies of K_r in G

Thm (Moon-Moser) $\forall$ n-vertex graph G,

$$\Rightarrow \quad \frac{k_{r+1}(G)}{k_r(G)} \geq \frac{1}{r^2-1}\left(r^2 \frac{k_r(G)}{k_{r-1}(G)} - n\right).$$

In particular, if $e(G) = \left(1 - \frac{1}{x}\right)\frac{n^2}{2}$ for some $x \in \mathbb{R}^+$

then $k_r(G) \geq \binom{x}{r}\left(\frac{n}{x}\right)^r$.

Pf • We prove the case $r=2$, i.e. $\frac{k_3(G)}{e(G)} \geq \frac{1}{3}\left(4\frac{e(G)}{n} - n\right)$

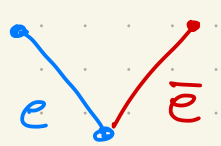
and leave the general case and "in particular" part as exercises.

• Let $e_1, \dots, e_m$ be all edges of G and $v_1, \dots, v_n$ be vxs of G.

Let $t_i = \#\, \Delta s$ containg e_i

and $d_i = d(v_i)$.

• We shall count



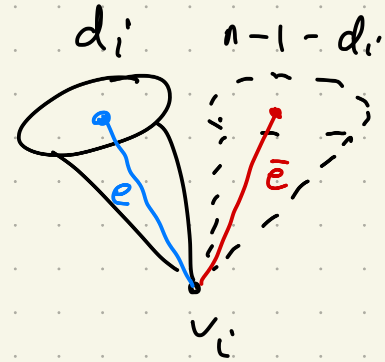
$P = \#$ pairs $(e, \bar{e})$ where

• $e \in E(G)$

• $\bar{e} \notin E(G)$

• $|e \cap \bar{e}| = 1$

- Note that $P = \sum_{i=1}^{n} d_i(n-1-d_i)$



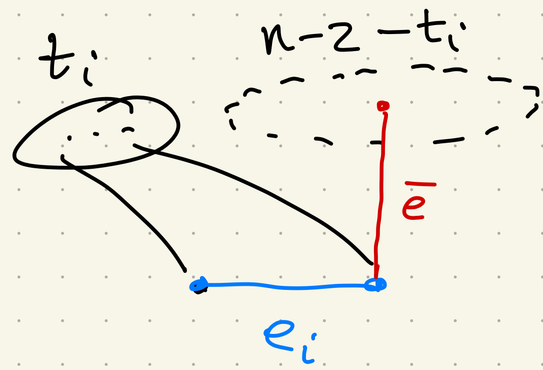
Jensen's ineq

$$\leq n \cdot \bar{d}\,(n-1-\bar{d})$$

$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{2m}{n}$

$$= 2(n-1)m - \frac{(2m)^2}{n}.$$

- $P \geq \sum_{i=1}^{m} (n-2-t_i)$



$\sum_{i=1}^{m} t_i = 3k_3(G)$

$$= m(n-2) - 3k_3(G)$$

Combining the lower & upper bounds on $P \Rightarrow$ 

Rmk $\forall x \in \mathbb{R}^+$

$$\binom{x}{r} := \frac{x(x-1)\cdots(x-r+1)}{r!} \quad \text{when } x > r-1$$

$$\text{o.w.} = 0 \quad (x \leq r-1)$$

In particular, if $e(G) = \left(1-\frac{1}{x}\right)\frac{n^2}{2}$ for some $x \in \mathbb{R}^+$

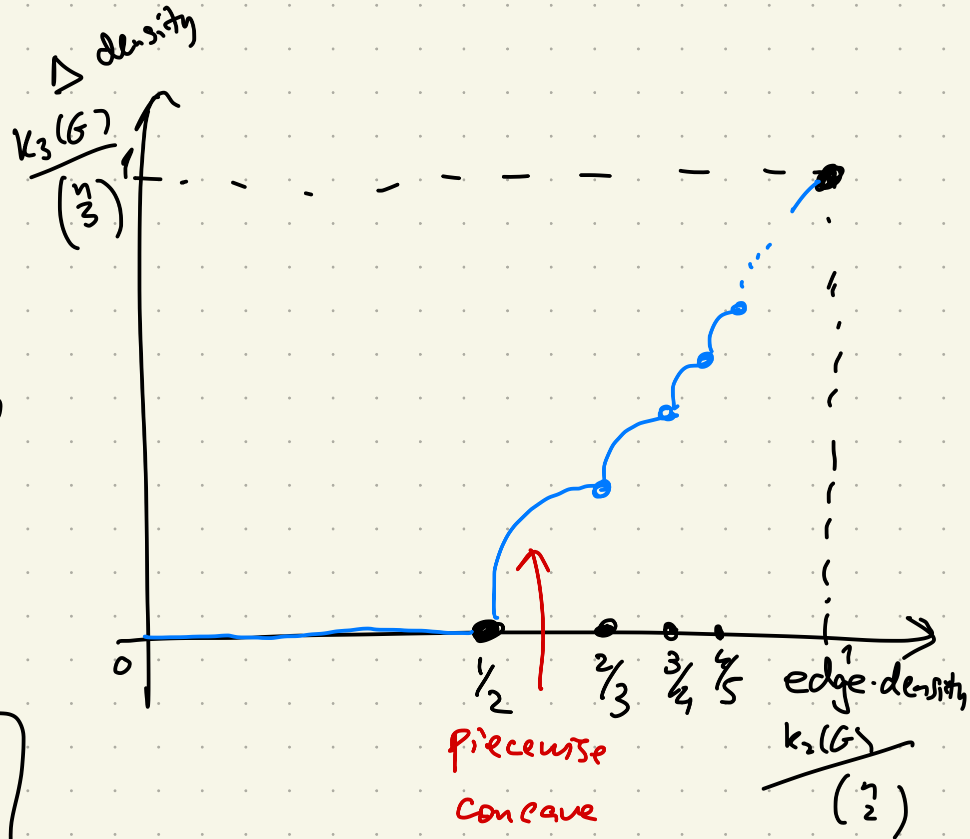
then $k_r(G) \geq \binom{x}{r}\left(\frac{n}{x}\right)^r$.

Moon-Moser gives a lower bound on clique density in a graph with given edge density

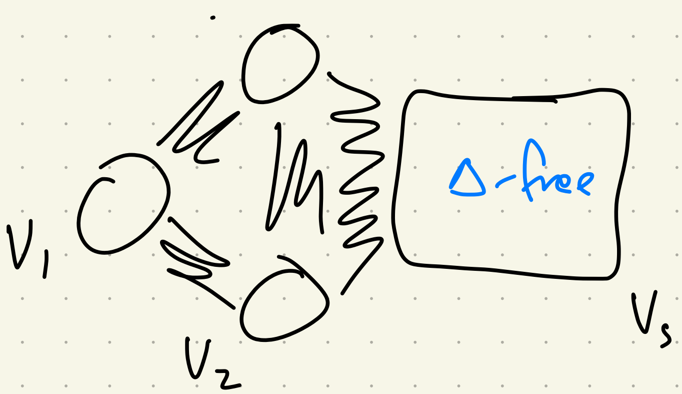
Q: Given edge density (or n vxs m edges)

what is the min # K_r's in a graph?

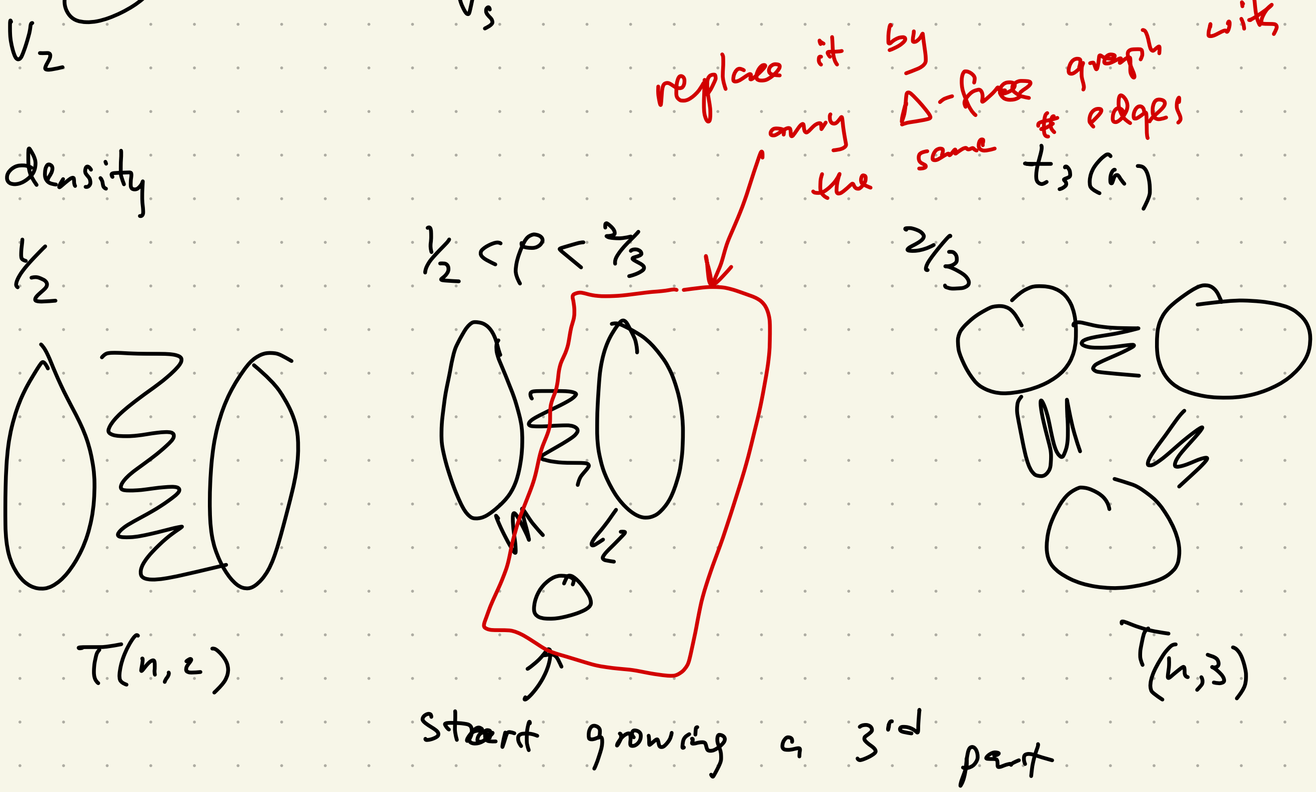
• Edges vs triangles



Conj Given an (n, e)-graph, one of the extremal graphs (w/ min # Δs) is of the following structure:



edge density



Clique density

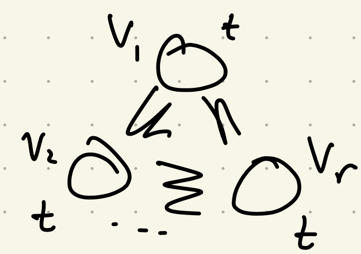
- Razborov 08 : K_3 case
- Nikiforov 11 : K_4
- Reiher 16 all cases.

Exact : L.-Pikhurko-Staden : K_3

Stability : Kim-L.-Pikhurko-Shaifeadeh all cases

Recall : ESS : # edges $\geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2}$

$\Rightarrow \quad \exists \quad K_r(t)$



$r = 3$

A careful calculation shows that

ESS $\Longrightarrow$ a copy of $K_r(t)$,

$t = \Omega(\log n)$

Rmk : $\log n$ is optimal

consider random graphs.

$e(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2} \xrightarrow{\text{supersaturation}}$ # K_r in G $\Omega(n^r)$ $\xrightarrow{\text{Nikiforov}}$ $K_r(t)$, $t = \Omega(\log n)$

$e(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2} \xrightarrow{\text{ESS}} K_r(t)$, $t = \Omega(\log n)$

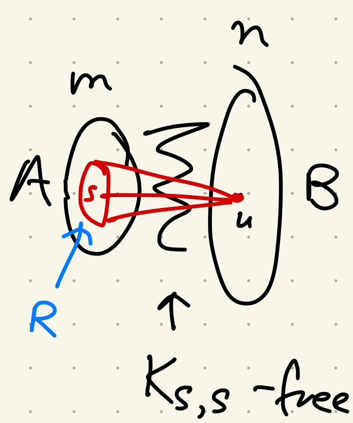
Thm (Nikiforov 07) Let $r \geq 2$ and G be an n-vertex graph w./ $\geq cn^r$ copies of K_r, where $c > (\log n)^{-1/r}$.

$\Rightarrow$ Then G contains a copy of $K_r(s)$, where $s = c^r \log n$.

Without specifying $\log n = \ln n$.

Lem (Asym. version of Kovari-Sós-Turán)

Let F be a bip. graph with partite sets A and B. Let $|A| = m$, $|B| = n$, $r \geq 2$, $(\log n)^{-1/r} < c < \frac{1}{2}$ and $s = c^r \log n$.



If $s \leq \frac{cm}{2}$ and $e(F) \geq cmn$

$\Rightarrow$ then F contains a $K_{s,s}$.

Suppose to the contrary that F is $K_{s,s}$-free.

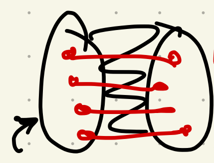
Pf: Count $N :=$ # s-star w./ center in B

$$(s-1)\binom{m}{s} \geq N = \sum_{u \in B} \binom{d(u)}{s}$$

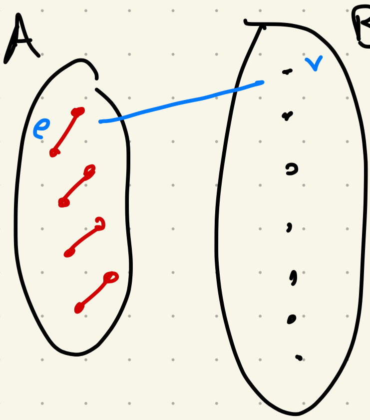
as F is $K_{s,s}$-free, each s-set R is counted at most $s-1$ times in this sum

------ calculation leads to a contradiction

Idea: Auxiliary bip. F



$M = A$ $\quad$ $B = V(G)$



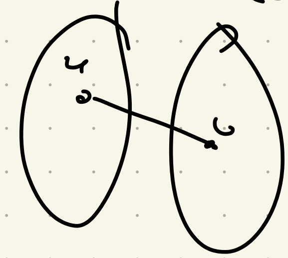
- $e \underset{F}{\sim} v \iff e, v$ form a Δ in F.
- $K_{\log n, \log n}$ in $F \implies K_{\log n, \log n, \log n}$ in G

Pf: We use induction on $r \geq 2$.

Base case: $r = 2$ $\quad$ $k_2(G) \geq cn^2$

Build an auxiliary graph F on partite sets $V(G), V(G)$

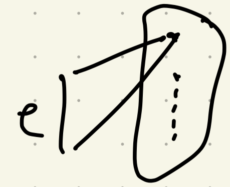
$u \in A \sim v \in B$ iff $uv \in K_2(G)$



$e(F) = 2K_2(G) > cn^2$

Lem $\implies$ $K_{s,s} \subseteq F \xrightarrow{\text{correspond}} K_{s,s} \subseteq G$

Inductive step $r=3$



Let $\mathcal{T} = K_3(G)$

If $\exists$ an edge e in $K_2(\mathcal{T})$ w/

$d_{\mathcal{T}}(e) = \#\Delta$s in $\mathcal{T}$ containing e $\leq cn$

then delete all Δs containing e from $\mathcal{T}$.

Claim $|\mathcal{T}| \geq cn^3/2$

Pf: $\#\Delta$s deleted in the above process

$$\leq k_2(G)\cdot cn \leq \frac{c}{2}n^3$$

- Note that $\forall\, e \in K_2(\mathcal{T})$ is contained in $\geq cn$ many Δs in $\mathcal{T}$

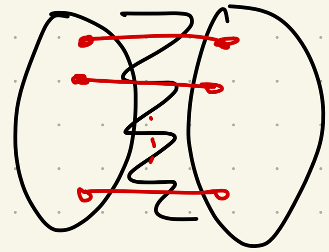
- Note that $|K_2(\mathcal{T})| \geq 3\cdot|\mathcal{T}|/n$

each Δ in $\mathcal{T}$ has 3 edges and each edge is counted $\leq n$ times

$$\geq \frac{3cn^3/2}{n} = \frac{3c}{2}n^2$$

- Lem $\Rightarrow$ $K_2(T)$ contains a $K_{m,m}$

  where $= \left(\frac{3c}{2}\right)^2 \log n$

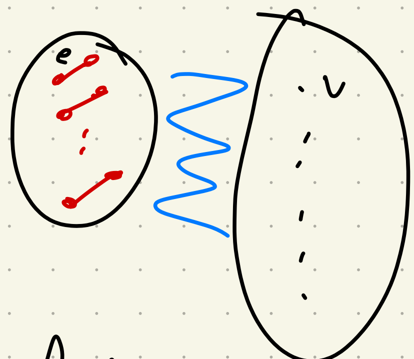


m m

- Build aux. bip F on partite set

$B=V(G)$ and $A =$ a perfect matching in $K_{m,m}$

$e \underset{F}{\sim} v$ iff e and v form a Δ in T.



A

m

$B=V(G)$

Recall $s = c^3 \log n$, want to find $K_{s,s,s}$

- $e(F) \geq m \cdot cn$

$\overset{\text{Lem}}{\Longrightarrow}$ F contains a $K_{s,s}$

which corresponds a $K_{s,s,s}$ in G.

- Shapira – Yuster
- Rödl – Schacht

Q: $\# \Delta s = \frac{n^3}{e^{c\sqrt{\log n}}}$

$\Longrightarrow$ $\Theta(\sqrt{\log n})$ -blowup of K_3 ?

- Fox-Luo-Wigderson (better quantitatively)