Lecture 5

Recall ESS: $ex(n,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\frac{n^2}{2}$

What happens when an n-vx graph G has more edges than this threshold?

- ESS $\Rightarrow$ $\exists$ one copy of H in any such G.
- In fact, we shall see that "many" copies emerge.

Such phenomenon is called supersaturation.

Def: Given a graph H, its Turán density is

$$\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$$

- ESS: $\forall H$, $\pi(H) = 1 - \frac{1}{\chi(H)-1}$.

Thm For any graph H, $\pi(H)$ exists.

Pf: • Let $\frac{ex(n,H)}{\binom{n}{2}} = a_n \leq 1$

Suffices to show that this bounded seq $\{a_n\}_{n\in\mathbb{N}}$ is non-increasing.

• Take an n-vx H-extremal graph G_n. So

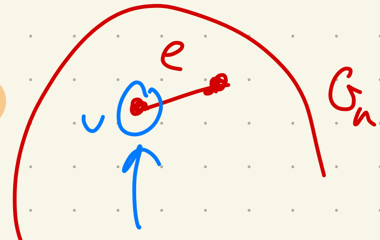
$$\frac{e(G_n)}{\binom{n}{2}} = a_n.$$

• Uniformly at random delete a vertex v from G_n and let the resulting (random) graph be G_{n-1}.

$$\mathbb{E}\left(e(G_{n-1})\right) = \left(1-\frac{2}{n}\right)\cdot e(G_n)$$

- By linearity of expectation
- for each edge $e \in E(G_n)$

$\Pr(e \text{ remains in } G_{n-1}) = 1 - 2/n$



$\Pr($the random deleted vx is an endpt of $e) = 2/n$

$$= \frac{n-2}{n}\cdot a_n \binom{n}{2}$$

$$= a_n \cdot \binom{n-1}{2}$$

• Therefore, $\exists$ a choice of v s.t. $G_{n-1} = G_n - v$ has $\geq a_n \binom{n-1}{2}$ edges.

As G_{n-1} is still H-free

$$a_n \leq \frac{e(G_{n-1})}{\binom{n-1}{2}} \leq \frac{ex(n-1,H)}{\binom{n-1}{2}} = a_{n-1}.$$

Rmk: • The same argument works for proving that Turán density for any r-unif hypergraph for all $r \in \mathbb{N}$

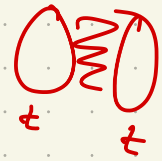
exists.

- ESS shows that the Turán density for a graph is of the form $1-\frac{1}{k}$ for some $k \in \mathbb{N}$.

For bip. H, $\pi(H)=0$ $\quad \pi(K_2(t))=0$

Def: blowup .....

$H \subseteq K_2(t) = K_{t,t}$



Erdős 64 $\quad \forall\, t \geq 1,\ r \geq 2$,

$$\pi(K_r^{(r)}(t)) = 0$$

§ Supersaturation

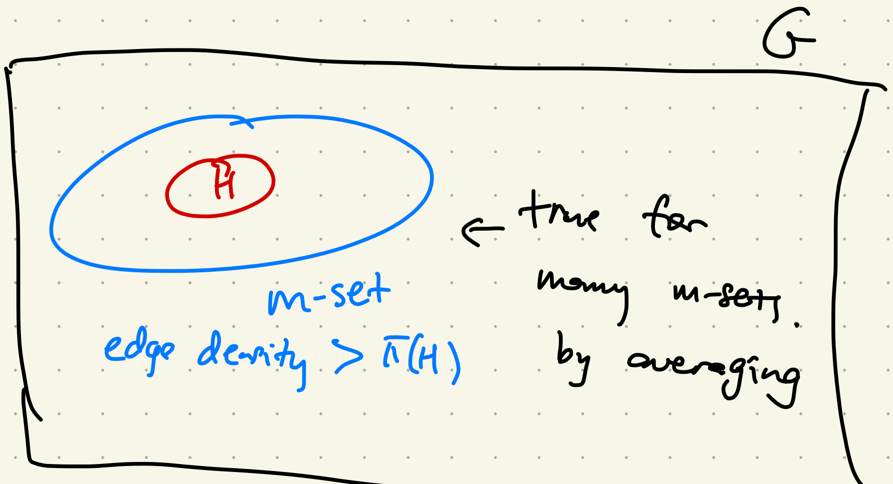
Thm Given $\varepsilon > 0$ and ~~a~~ an h-vertex graph H, there exist $\delta > 0$ and $n_0 > 0$ s.t. T.F.H. for all $n \geq n_0$.

If G is an n-vertex graph w/ $e(G) \geq (\pi(H)+\varepsilon)\binom{n}{2}$

$\Rightarrow$ then G contains $\geq \delta \binom{n}{h}$ copies of H.

Pf: • Take a suff. large m s.t.

$$ex(m, H) \leq \left(\pi(H) + \frac{\varepsilon}{4}\right)\binom{m}{2}$$



We claim that $\geq \frac{1}{4}\varepsilon\binom{n}{m}$ m-sets in $V(G)$, each inducing a subgraph w/ $\geq \left(\pi(H) + \varepsilon/4\right)\binom{m}{2}$ edges.

Indeed, otherwise

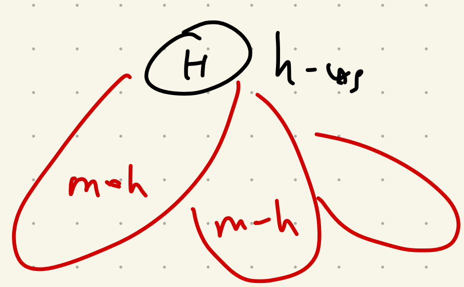
$$\sum_{M \in \binom{V(G)}{m}} e(G[M]) \leq \overbrace{\tfrac{1}{4}\varepsilon\binom{n}{m}}^{\text{m-sets w./ many edges}} \cdot \binom{m}{2} + \overbrace{\binom{n}{m}}^{\text{m-sets w./ few edges}} \cdot \left(\pi(H) + \tfrac{\varepsilon}{4}\right)\binom{m}{2}$$

$$= \left(\pi(H) + \varepsilon/2\right)\binom{n}{m}\binom{m}{2}$$

$$= \binom{n-2}{m-2} \cdot e(G) \geq \left(\pi(H) + \varepsilon\right)\binom{n}{2}\binom{n-2}{m-2}$$

$$= \left(\pi(H) + \varepsilon\right)\binom{n}{m}\binom{m}{2} \quad \lightning$$

- By def, each of these m-set contains a copy of H, and each copy of H is contained in $\leq \binom{n-h}{m-h}$ m-sets



$\Longrightarrow$ # copies of H in G

$$\geq \frac{\frac{1}{4}\varepsilon\binom{n}{m}}{\binom{n-h}{m-h}} \geq \underbrace{\frac{\varepsilon}{4\binom{m}{h}}}_{:=\delta} \cdot \binom{n}{h}.$$

The Turán density of a graph is the same as its blowups.

Prop $\forall\, r, t \in \mathbb{N}$, $\pi(K_r) = \pi(K_r(t))$

Pf: "$\leq$" trivial

"$\geq$" **NTS**: Fix $\varepsilon > 0$

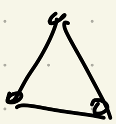
$$ex(n, K_r(t)) \leq (\pi(K_r) + \varepsilon)\binom{n}{2}$$

Let G be an n-vx graph w/ $e(G) \geq (\pi(K_r)+\varepsilon)\binom{n}{2}$.

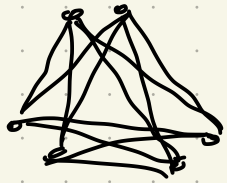
NTF $K_r(t) \subseteq G$.

$r=3$ $\quad t=2$

K_r



$K_3(2)$



$K_{2,2,2}$

Erdős 64 $\forall\, t \geq 1,\ r \geq 2$,

$\pi(K_r^{(r)}(t)) = 0$

- By supersaturation, G has $\geq \delta\binom{n}{r}$ copies of K_r

Build an auxiliary r-unif hyp. $\mathcal{H}$ on $V(G)$ the same vx set

where each copy of K_r in G is a hyperedge in $\mathcal{H}$

$$\Rightarrow e(\mathcal{H}) = \#K_r\text{'s in } G \geq \delta\binom{n}{r}$$

- Erdős $\Rightarrow$ $\exists$ a copy of $K_r^{(r)}(t)$ in $\mathcal{H}$

$\pi(K_r^{(r)}(t)) = 0$

This $K_r^{(r)}(t)$ in $\mathcal{H}$ corresponds to a copy of $K_r(t)$ in G

We obtain another pf of ESS via supersaturation.

<u>Pf of ESS</u>: Fix H w/ $\chi(H) = r$,

it suffices to show that $\pi(H) = \pi(K_r)$

as $\pi(K_r) = 1 - \frac{1}{r-1}$ by Turán's thm.

Note that $H \subseteq K_r(t)$, $t = |H|$,

so $\pi(H) \le \pi(K_r(t)) \overset{\text{prop}}{=} \pi(K_r)$.

$\pi(H) \ge \pi(K_r)$ by considering $\underbrace{T_{n,r-1}}_{H\text{-free}}$.

A graph G is (K_3, K_3)-free if edges of G can be 2-colored w/ no mono χ. Δs. Let

$$ex(n, K_3, K_3) = \max e(G) : \begin{array}{l} |G| = n \\ G \text{ is } (K_3, K_3)\text{-free.} \end{array}$$

Ex1: • Determine $ex(n, K_3, K_3)$.

• Determine $ex(n, K_{r_1}, \ldots, K_{r_t})$ (in terms of Ramsey #)

Ex2

We know $ex(n, K_r) + 1$ edges implies a copy of K_r. Show that it in facts implies a copy of K_{r+1}^- (K_{r+1} w./ an edge removed)

Rmk Regarding Ex 2, there are two extensions.

1) color-critical H

H is color-critical if $\exists\, e \in E(H)$ s.t. removing this edge reduces

$ex(n,H) = ex(n, K_{\chi(H)})$

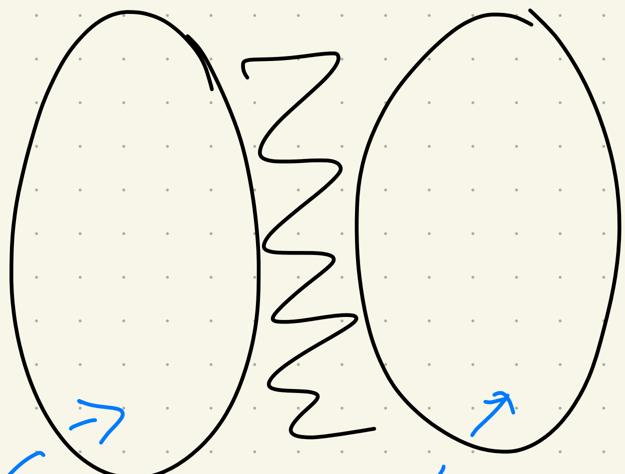
the chromatic #, i.e. $\chi(H-e) \leq \chi(H)-1$

K_{r+1}^{-} is a color-critical graph w./ chr. # r.

2) $ex(n, K_{2,2,t}) = \frac{n^2}{4} + 2\cdot ex(\frac{n}{2}, K_{2,2})$

Bollobás - Erdős

- Simonovits - Szemerédi



$n/2$ $\quad$ $n/2$

$n/2$-vertex $K_{2,2}$-extremal graphs