Lecture 4

§ Multicolor extension of symm. by Füredi - Maleki

Thm (F-M) Let G be a graph and $G_1, \dots, G_d \subseteq G$ be subgraphs, all on vertex set $[n]$.

For every $\underline{x} \in \Delta^{n-1}$, $\exists$ a set $K \subseteq V(G)$ and $\underline{y} \in \Delta^{n-1}$ with support K s.t.

- $y^T A_{G_i} y \geq x^T A_{G_i} x$, $\forall i \in [d]$
- $\alpha(G[K]) \leq d$

<u>Pf</u>: • Take $\underline{y} \in \Delta^{n-1}$ w/ minimal supp. and $y^T A_{G_i} y \geq x^T A_{G_i} x$ $\forall i \in [d]$.

and let $K = \mathrm{supp}(\underline{y})$

• Suppose $\{1, \dots, d+1\} \subseteq K$ is an indep set. (w.l.o.g.

$$\Rightarrow \forall\ z = (\underbrace{z_1, z_2, \dots, z_{d+1}}_{z' \in \mathbb{R}^{d+1}}, 0, \dots, 0)^T \in \mathbb{R}^n$$

$$\forall i \in [d]: \ z^T A_{G_i} z = \sum_{s \sim_{G_i} t} z_s z_t = 0$$

Define $a_i' \in \mathbb{R}^{d+1}$ where a_i' is the first $(d+1)$ coord. of a_i

$$\underline{a_i} = 2 A_{G_i} y$$

• $\forall \alpha \in \mathbb{R}$,

$$(\underline{y} + \alpha \underline{z})^T A_{G_i} (\underline{y} + \alpha \underline{z}) = \underline{y}^T A_{G_i} \underline{y} + 2\alpha \underline{z}^T A_{G_i} \underline{y}$$

$$= y^T A_{G_i} y + \alpha \cdot (a_i^T \cdot z) = y^T A_{G_i} y + \alpha ((a_i')^T \cdot z')$$

<u>Claim</u> $\exists$ non-zero vector $z' \in \mathbb{R}^{d+1}$ s.t.

- $a_i' \cdot z' \geq 0 \quad \forall i \in [d]$ and
- $z' \cdot 1 = 0$, i.e. $\sum_{i=1}^{d+1} z_i' = 0$

<u>Pf</u>: ~~# variables $= d+1$ $\geq$ # constraints $= d+1$ $\Rightarrow$ non-zero sol$^{\underline{n}}$ z'.~~ $\square$

- Claim $\Rightarrow$ Thm: by choosing appropriate $\alpha > 0$ we can make sure that $\mathrm{supp}(y + \alpha z) \subsetneq \mathrm{supp}(y)$ and $(y+\alpha z)^T A_{G_i} (y + \alpha z) \geq y^T A_{G_i} y \quad \forall i \in [d]$

$\lightning$ minimality of y.

<u>Pf of Claim</u> Let A be the $(d+1)\times(d+1)$ matrix

$$A = \begin{pmatrix} - & a_1' & - \\ & \vdots & \\ - & a_d' & - \\ 1 & 1 & 1 \; 1 \; 1 \; 1 \end{pmatrix}.$$

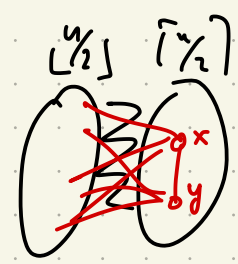
If $\det A = 0 \Rightarrow$ # var. $= d+1 \underset{d \geq}{>}$ # constraints

If $\det A \neq 0$, $Az = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ has a unique sol$^{\underline{n}}$, which is clearly non-zero. $\square$

- Erdős - Faudree - Rousseau '92

$\forall$ n-vx. G w/ $\frac{n^2}{4} + 1$ edges

$\Rightarrow \geq 2\lfloor n/2 \rfloor + 1$ edges in triangles



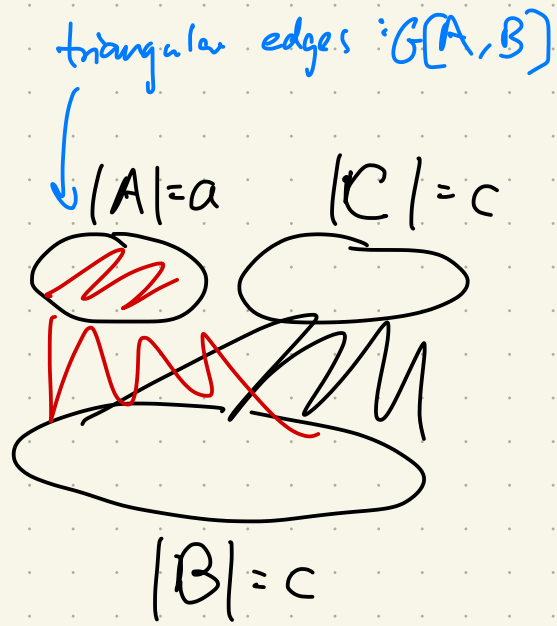
Def: $Tr(n,e) := \min\{$ edges in a triange in an n-vx e-edge graph $\}$

M: $Tr(n, n^2/4) = 0$

EFR: $Tr(n, n^2/4 + 1) = 2\lfloor n/2 \rfloor + 1$

F-M: Given n, $n^2/4 < e \leq \binom{n}{2}$

$a+b+c = n$



FM: $Tr(n,e) \geq$

$$\min\left\{ e - bc : \begin{array}{l} a+b+c = n \\ a,b,c \in \mathbb{R}^+ \\ \frac{1}{2}a^2 + ab + bc \geq e \end{array} \right\}.$$

- Gruslys - Letzter : exact version min C_3-edges
- Grzesik - Hu - Volec : min C_{2k+1}-edges $k \geq 2$

when G has n vxs & $\frac{n^2}{4}+1$ edges.

§ Erdős - Simonovits - Stone.

$\forall \varepsilon, H \ \exists n \geq n_0(\varepsilon, H)$ s.t.

$\left(1 - \frac{1}{\chi(H)-1} - \varepsilon\right)\frac{n^2}{2} \leq ex(n,H) \leq \left(1 - \frac{1}{\chi(H)-1} + \varepsilon\right)\frac{n^2}{2}$

<u>Thm</u> (ESS 46) $\forall H$,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\frac{n^2}{2}$$

<u>Rmk</u>: • If H is nonbipartite, i.e. $\chi(H) \geq 3$

then ESS gives the asymp. of the ext. # of H.

• If H is bipartite $\Rightarrow$ $o(n^2)$

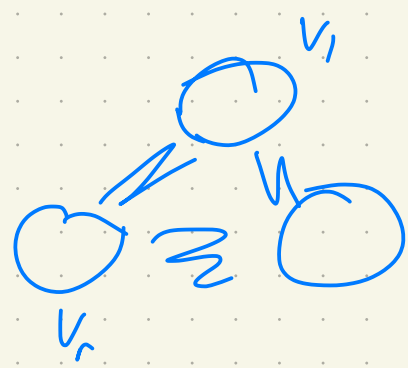
ESS doesn't say much about $ex(n, H)$.

degenerate case

$\chi(H) = r+1$, $ex(n,H) \geq \left(1 - \frac{1}{r} + o(1)\right)\frac{n^2}{2}$

Lower bound of ESS.

Consider $T_{n,r} \not\supseteq H$.



<u>Lem</u>: $\forall\ 0 < \varepsilon < c < 1$, $\exists\ n_0 = n_0(\varepsilon, c)$ s.t. $\forall n \geq n_0$

T.F.H. Let G be an n-vx graph w/ $e(G) \geq cn^2/2$

$\Rightarrow \exists$ subgraph $G' \subseteq G$ on $n' \geq \sqrt{\varepsilon}\, n/2$ vertices

and $\delta(G') \geq (c-\varepsilon)n'$.

<u>Pf</u>: • Let $G_n = G$ and construct a seq of graphs $G_n, G_{n-1}, \dots$ as follows

Given G_i, if $\delta(G_i) \geq (c-\varepsilon)n_i$ ☺

STOP

o.w. delete a vertex of min. deg from $G_i \Rightarrow G_{i-1}$

Let G' be the resulting graph.

$$\frac{cn^2}{2} \leq e(G) \leq e(G') + \sum_{i=n'+1}^{n} (c-\varepsilon)\cdot i$$

$$\leq \frac{(n')^2}{2} + (c-\varepsilon)\sum_{i=n'+1}^{n} i$$

$\cdots\cdots \Rightarrow n' \geq \frac{\sqrt{\varepsilon}}{2}\cdot n$

By construction, $\delta(G') \geq (c-\varepsilon)n'$. ▨

Let $r = \chi(H) - 1$

<u>Pf</u> (Erdős - Simonovits - Stone)

Given G n-vx w/ $\left(1 - \frac{1}{r} + 2\varepsilon\right)\frac{n^2}{2}$ edges

we need to show $H \subseteq G$.

- Lem $\Rightarrow$ $G' \subseteq G$ on $n' \geq \frac{\sqrt{\varepsilon}}{2} n$ vxs (so n' is suff. large when n is suff. large)

  and $\delta(G') \geq \left(1 - \frac{1}{r} + \varepsilon\right) n'$.

Let $t = |H|$, suffices to embed $K_{\underbrace{t, \dots, t}_{r+1 \text{ parts}}}$ in G'.

We shall prove by induction on $1 \leq s \leq r+1$

that $\forall T$, $n' \geq n_0(T)$, then

$\exists$ $K_{\underbrace{T, \dots, T}_{s \text{ parts}}}$.

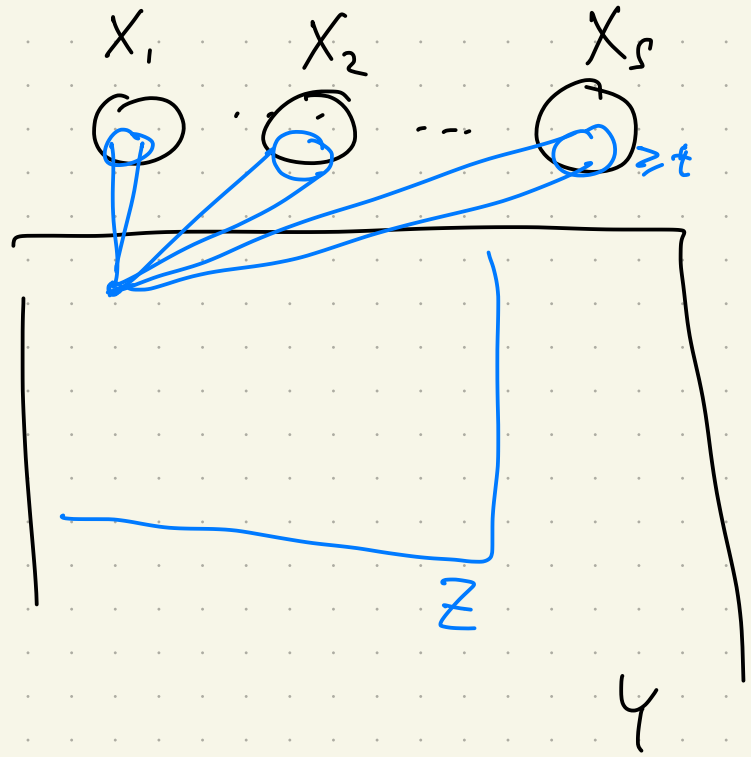
Base case: $s = 1$, trivially true

Inductive step: by induction hypothesis

$\exists$ $K_{\underbrace{T, \dots, T}_{s \text{ parts}}}$ for $s \leq r$, where $T = \frac{4}{r \cdot \varepsilon} \cdot t$

w/ partite set $X_1, \dots, X_s$.

Let $Y = V(G) \setminus \left(\bigcup_{i=1}^{s} X_i\right)$



<u>Claim</u>: Let $Z \subseteq Y$ be the set of all $u \in s$ w/ $\geq$ $\frac{r\varepsilon}{4}|X_i| = t$ neighbors in each X_i, $i \in [s]$.

Then $|Z| \geq \frac{r\varepsilon}{4} n'$

- Claim $\Rightarrow$ thm: each vertex in Z gives rise to a copy of $K_{1,\underbrace{t,\dots,t}_{s \text{ parts}}}$ w/ the size-t parts in $X_1, \dots, X_s$ respectively.

$$\#\text{ choices for } t\text{-subsets } X_i' \subseteq X_i \leq \binom{T}{t}^s$$

$$\Rightarrow \text{ pigeonhole } \geq \frac{|Z|}{\binom{T}{t}^s} \geq \frac{r\varepsilon}{4\binom{T}{t}^s} n \geq t$$

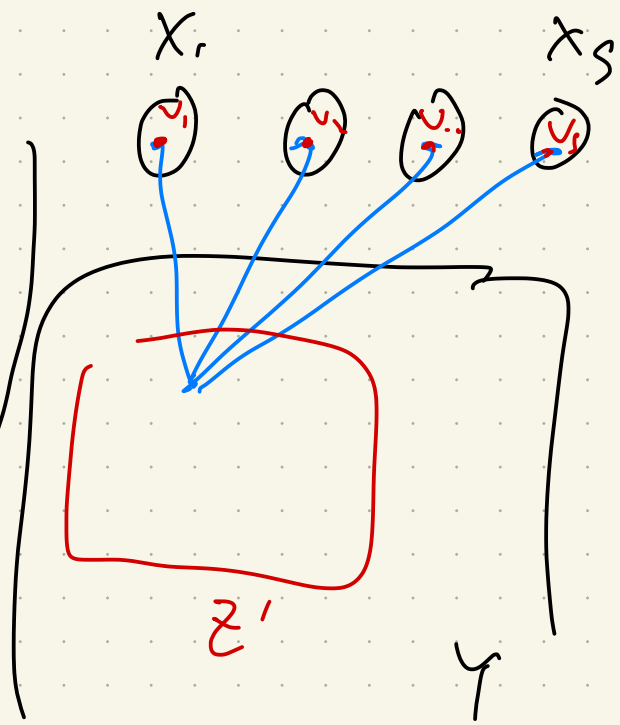
vertices ...

Pf (Claim) Let us first bound # s-star N with center in Y and exactly 1 leaf in each X_i

For each $(v_1, \dots, v_s) \in (X_1, \dots, X_s)$

$$\geq r\varepsilon n' - \left|\bigcup_{i=1}^{s} X_i\right|$$

$$\geq r\varepsilon n'/2 \quad \text{common neighbors in } Y.$$



$\delta(v_i) \geq (1 - \frac{1}{r} + \varepsilon) n'$

$s \leq r$

$\Rightarrow$ # common neighbors of $v_1, \dots, v_s \geq r\varepsilon n'$

$$\Rightarrow \quad N \geq \prod_{i=1}^{s} |X_i| \cdot r\varepsilon n'/2 \qquad (1)$$

Let $Z' \subseteq Y$ be the set of vxs involved in $\geq \frac{r\varepsilon}{4} \prod_{i=1}^{s} |X_i|$ many such s-star

Note that, by defn, each vx in Z' has $\geq \frac{r\varepsilon}{4} |X_i|$ neighbors in X_i $\forall i \in [s]$

Thus $Z' \subseteq Z$.

- $$N \leq |Z'| \cdot \prod_{i=1}^{s} |X_i| + n' \cdot \frac{r\varepsilon}{4} \prod_{i=1}^{s} |X_i| \quad \cdots (2)$$

(1) & (2) $\Rightarrow |Z'| \geq \frac{r\varepsilon}{4} n'$. $\blacksquare$