Lecture 3

§ Applications of symm. method

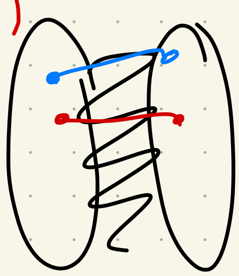
§ Erdős-Rothschild problem. (1974)

Consider n-vertex graph G, how many 2-edge-colorings $F(G;3,3)$ "#" (not necessarily proper coloring) can it have without monochromatic Δs? They want to find the max. # of such colorings among all n-vertex graphs.

$$F(n;3,3) = \max_{|G|=n} F(G;3,3)$$

A simple lower bound: consider $G = T_{n,2} = K_{n/2, n/2}$

$$F(n;3,3) \geq 2^{n^2/4} = 2^{\underbrace{t_2(n)}} \to e(T_{n,2})$$



Full setup of Erdős-Rothschild Problem:

- $\underline{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$
- A edge-coloring of G (not necessarily proper) is $\underline{k}$-valid

if there is no K_{k_i} in i^{th} color $\forall i \in [s]$.

- $F(G;\underline{k}) = \#$ $\underline{k}$-valid colorings of G

$$F(n;\underline{k}) = \max_{|G|=n} F(G;\underline{k})$$

When all $k_i = k$,

$$F(n;\underbrace{(k,k,\dots,k)}_{s \text{ colors}}) \geq s^{t_{k-1}(n)} \quad \cdots\cdots \quad (L)$$

<u>Thm</u> (Alon-Balogh-Keevash-Sudakov)

Let $k, n \in \mathbb{N}$, where $k \geq 3$ and $n \geq n_0(k)$.

Then $F(n;(k,k)) = 2^{t_{k-1}(n)}$, $F(n;(k,k,k)) = 3^{t_{k-1}(n)}$,

and $T_{k-1}(n)$ is the unique extremal graph for both cases.

However, when $s \geq 4$ (at least 4 colors), it is known that $F(n,\underline{k})$ is exp. larger than (L).

<u>Thm</u> (Pikhurko-Staden) For $\underline{k} = \overbrace{(3,\dots,3)}^{7 \text{ colors}}$,

$T_{n,8}$ is the unique extremal graph for $F(n;\underline{k})$ w/ colorings coming from Hadamard matrices of order 8.

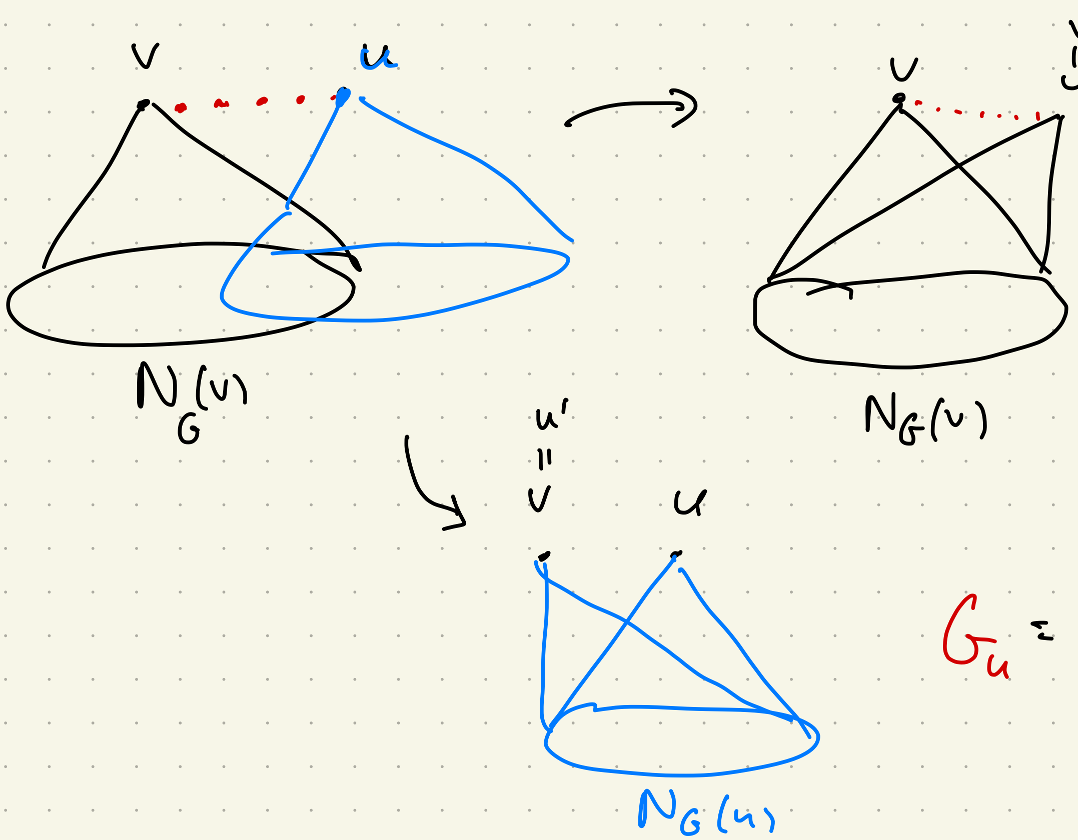
Using Zykov's symm. technique,

Thm (Pikhurko - Staden - Yilma)

For every $n, s \in \mathbb{N}$, and $\underline{k} \in \mathbb{N}^s$, at least one of the $\underline{k}$-extremal graphs of order n is complete multipartite.

Pf: • Take an n-vertex $\underline{k}$-extremal graph G.

Consider a non-adjacent pair of vxs $u, v \in V(G)$



$G_v = G - u + v'$

$G_u = G - v + u'$

• For a graph H, let $\mathcal{F}(H)$ be the set of all $\underline{k}$-valid colorings of H. (so $F(H; \underline{k}) = |\mathcal{F}(H)|$)

• Let $G' = G - u - v$ and $\sigma \in \mathcal{F}(G')$, and let σ_u (resp. σ_v) be the number of $\underline{k}$-valid extensions of σ to $G - v$ (resp. $G - u$)

• Observe that, as $uv \notin E(G)$, and each forbidden config. is a clique,

$\Rightarrow$ # $\underline{k}$-valid extensions of σ to G is $\sigma_u \cdot \sigma_v$.

$$F(G;\underline{k}) = \sum_{\sigma \in \mathcal{F}(G')} \sigma_u \cdot \sigma_v$$

On the other hand

$$F(G_u;\underline{k}) = \sum_{\sigma \in \mathcal{F}(G')} \sigma_u^2 \quad \text{and}$$

$$F(G_v;\underline{k}) = \sum_{\sigma \in \mathcal{F}(G')} \sigma_v^2$$

• As G is $\underline{k}$-extremal

$\Rightarrow$
$$0 \leq 2F(G;\underline{k}) - F(G_u;\underline{k}) - F(G_v;\underline{k})$$

$$= \sum_{\sigma \in \mathcal{F}(G')} (2\sigma_u\sigma_v - \sigma_u^2 - \sigma_v^2)$$

$$= \sum_{\dots} -(\sigma_u - \sigma_v)^2 \leq 0$$

Thus "=" must hold above and G_u and G_v are also $\underline{k}$-extremal graph.

Therefore, we can keep operating Symm. as in Zyka's pf of Turán's thm and obtained at the end a complete multipartite $\underline{k}$-extremal graph. $\blacksquare$

§ Weighted Turán

<u>Thm</u>$^{(*)}$ (Bradač; Malec – Tompkins)

Let G be an n-vertex graph and for each edge $e \in E(G)$, let $c(e)$ be the size of largest clique in G containing e.

$$\Longrightarrow \sum_{e \in E(G)} \frac{c(e)}{c(e)-1} \leq \frac{n^2}{2}$$

<u>Ex.</u> Derive Turán's thm from this weighted version.

Ex. Prove Thm (*) using M-S symm.

Ex. Use induction on # vxs n to prove Thm (*) (Similar to pf 1 of Turán).

Related results:

- Füredi - Kündgen (01)
- Zixiang Xu - Yifan Jing - Gennian Ge (22)

- $\underline{v} \in \mathbb{R}^n$, ℓ_p-norm: $\|v\|_p = \left(\sum_{i\in[n]} v_i^p\right)^{1/p}$

- For an $n\times n$ real symm. matrix A,

its Frobenius norm (Hilbert-Schmidt norm) is

$$\|A\|_F = \sqrt{\sum_{i,j\in[n]} A_{i,j}^2}$$

$$= \sqrt{tr(A^2)} = \sqrt{\lambda_1^2+\lambda_2^2+\dots+\lambda_n^2},$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are eigenvalues of A.

Example Take $A = A_G$ adj. matr. of a graph

G: on vertex set $[n]$

$$\mathrm{tr}(A^2) \overset{\text{Ex}}{=} \sum_{i \in [n]} d(i) = 2e(G)$$

$$\Rightarrow \quad \|A\|_F = \sqrt{2e(G)}.$$

- By defn: $\lambda_1^2 \leq \|A\|_F^2$

When $A = A_G$, G is bipartite

$$\Rightarrow \quad \lambda_n = -\lambda_1$$

$$\Rightarrow \quad \lambda_1^2 \leq \frac{1}{2}\|A\|_F^2$$

Prop (Ganguly - Nam) Let G be a graph with clique # $\omega(G) = r$ and $A = A_G$ be its adj. mat. $\Rightarrow$

$$\lambda_1(A)^2 \leq \frac{r-1}{r}\|A\|_F^2$$

$\underline{\text{Ex}}$ Show that $\Uparrow$ tight for K_r.

$\underline{\text{Pf}}$:

- Courant-Fisher

$$\lambda_1 = \sup_{\|x\|_2 = 1} x^T A x = \sup_{\|x\|_2=1} \sum_{i,j \in [n] : i \sim j} x_i x_j$$

- $$\frac{\lambda_1^2}{\|A\|_F^2} = \sup_{\|x\|_2=1} \frac{\left(\sum_{i \sim j} x_i x_j\right)^2}{2e(G)} \overset{C-S}{\leq} \sup_{\|x\|_2=1} \frac{\cancel{2e(G)} \cdot \sum_{i \sim j} x_i^2 x_j^2}{\cancel{2e(G)}}$$

Let $y \in \mathbb{R}^n$ be s.t. $y_i = x_i^2$

$$\|x\|_2 = 1 \iff \sum_{i \in [n]} x_i^2 = 1 = \sum_{i \in [n]} y_i \Rightarrow \|y\|_1 = 1$$

$$= \sup_{\|y\|_1 = 1,\ y_i \geq 0} \sum_{i \sim j} y_i y_j = \sup_{y \in \Delta^{n-1}} y^T A y$$

(i.e. $y \in \Delta^{n-1}$)

$$\overset{M-S}{\leq} \frac{r-1}{r}.$$

$\blacksquare$