Lecture 11

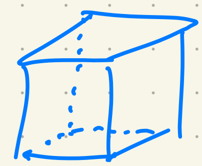
- $\delta_\chi(K_3) = 1/3$, Hajnal's constr. $\geq 1/3$

  Today $\leq 1/3$

<u>Thm</u> (Łuczak - Thomassé 10)

$\forall$ n-vx Δ-free G w/ $\delta(G) > n/3$ has $\chi(G) \leq 1665$.

We need the following structural result.

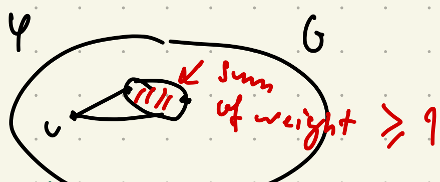


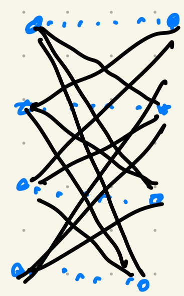
<u>Thm</u> (Brandt) G • n-vx $\delta(G) > n/3$ • maximal Δ-free $\Longrightarrow$ no induced Q_3, (3-dim. cube)

<u>Def</u>: A weight funct. $\varphi: V(G) \to \mathbb{R}_{\geq 0}$

$Q_3 = K_{4,4} \setminus PM$

is feasible if $\forall v \in V(G)$, $\varphi(N(v)) := \sum_{u \in N(v)} \varphi(u) \geq 1$





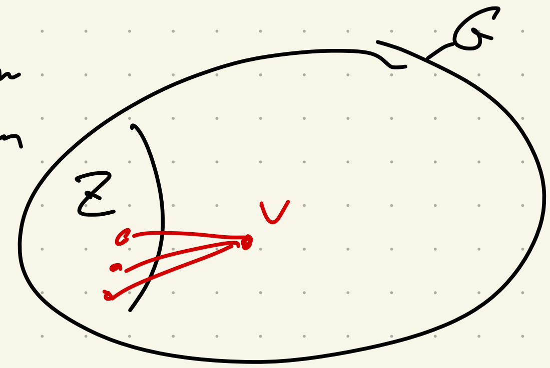
E.g. $\varphi \equiv \frac{1}{\delta(G)}$ is feasible.

<u>Lem</u> • G maximal Δ-free • φ feasible w/ $\varphi(V(G)) < 3$ $\Longrightarrow$ no induced Q_3.

[Lem $\Rightarrow$ Thm]: Consider $\varphi(v) = 1/\delta(G)$ feasible, $\varphi(V(G)) = \frac{n}{\delta(G)} < 3$ $\boxtimes$

Prop $\forall\, Z \subseteq V(G)$, $\forall\, \varphi$ feasible, let D be the maximum # of neighbr. a vx can have in Z

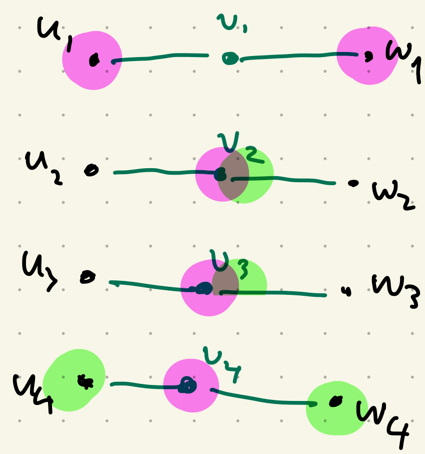
$$D := \max_{v \in V(G)} d(v, Z)$$



$$\Rightarrow \quad D \geq \frac{|Z|}{\varphi(V(G))}$$

Pf: $$|Z| \overset{\text{feasible}}{\leq} \sum_{v \in Z} \varphi(N(v)) = \sum_{v \in V(G)} d(v, Z)\, \varphi(v) \leq D \cdot \varphi(V(G)). \quad \square$$

Pf (Lem) Supp. $\exists$ an induced copy of Q_3, say w/ part. sets $\{u_1, \dots, u_4\}$ and $\{w_1, \dots, w_4\}$ and $u_i w_i \notin E(G)$.



- maximality of $G \Rightarrow$ diameter ≤ 2

$\Rightarrow \exists\, v_i \in N(u_i) \cap N(w_i)$

Δ-free $\Rightarrow v_i \not\sim u_j, w_j \ \forall j \neq i$.

Claim $\exists\, x$ w/ deg > 4 to $\{u_i, w_i, v_i\}_{i \in [4]} = Z$

Pf: Supp. not, $D := \max_v d(v, Z) \leq 4$

Prop $\Rightarrow D \geq \dfrac{|Z|}{\varphi(V(G))} > \dfrac{12}{3} = 4$. $\lightning$. $\square$

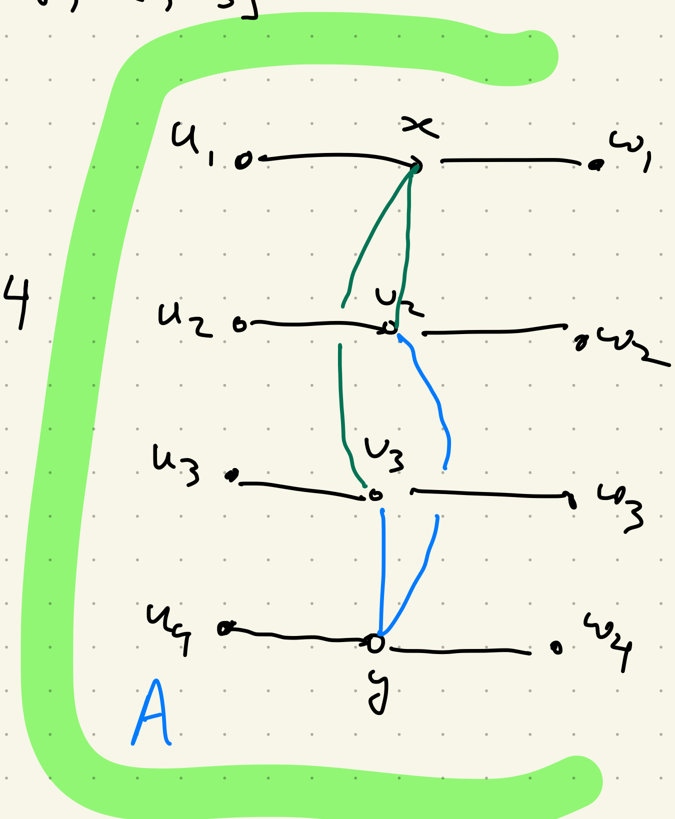
W.l.o.g. $x \sim u_1, w_1, v_2, v_3, v_4$

Similarly, $\exists\, y$ w/ deg > 3 to $\{u_i, w_i, v_i\}_{2 \leq i \leq 4}$

W.l.o.g. $y \sim u_4, w_4, v_2, v_3$

Δ-free $\Rightarrow x \neq y$. , $y = x, w_1, u_4 \cong \Delta$

- Consider $\{u_i, w_i\}_{i \in [4]} \cup \{x, y, v_2, v_3\}$

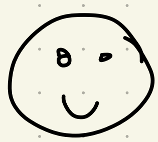


- Prop to A

$\Rightarrow \max_v d(v, A) \geq \dfrac{|A|}{\varphi(V(G))} > \dfrac{12}{3} = 4$

i.e. $\exists$ a vx having ≥ 5 neighbrs in A. $\Rightarrow \exists \Delta$

as $\alpha(G[A]) = 4$

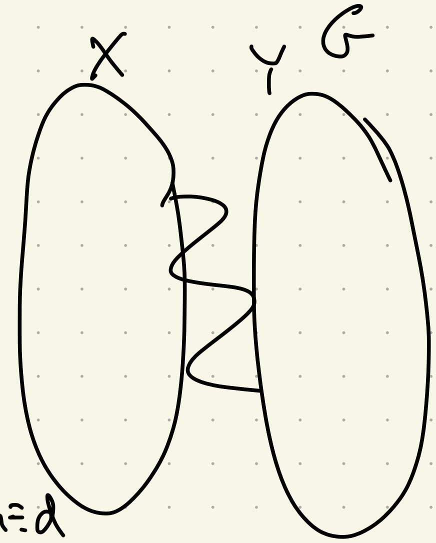


Thm2 (Ł-T) $\forall$ n-vx Δ-free G w./ min deg cn and no induced Q_3 $\Rightarrow$

$$\chi(G) \leq \frac{192 \log \frac{6}{c}}{c}$$

With the struct. result of Brandt, the above thm $\Rightarrow$

$$\delta_\chi(G) \leq 1/3.$$



Max cut

Idea Take a max cut X, Y of G.

1) Consider the neighborhood hyp. induced by X

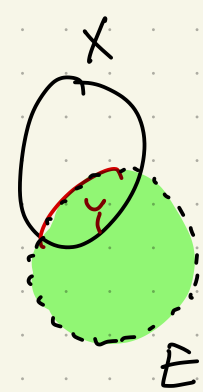
$$N_X = (Y, \{N(v) : v \in X\})$$

induced Q_3-free $\Rightarrow$ N_X has bdd VC dim $= d$

ε-net thm $\Rightarrow$ $\tau(N_X) \leq f(\tau^*(N_X), d)$

- $\chi(G[X]) \leq \tau(N_X)$
- min deg $= \Omega(n)$ $\Longrightarrow$ N_X has bdd fract. transversal #.

  max cut



Def: A set X is **shattered** by $\mathcal{F}$ if

$$\forall\, Y \subseteq X, \quad \exists\, E \in \mathcal{F} \text{ s.t. } E \cap X = Y$$

The VC dim of $\mathcal{F}$ is the size of a set shattered by $\mathcal{F}$. That is,

$$d_{VC}(\mathcal{F}) = \max r \in \mathbb{N} : \exists\, |X| = r \text{ s.t. } \mathcal{F}\big|_X = 2^X.$$

Def: The **transversal #** of $\mathcal{F}$ is the cardinality of a smallest set T s.t. T intersects all edges of $\mathcal{F}$.

$$\tau(\mathcal{F})$$

Def: A fractional transversal $\varphi : V(\mathcal{F}) \to [0,1]$ of $\mathcal{F}$ satisfies $\forall E \in \mathcal{F}$, $\sum_{v \in E} \varphi(v) \geq 1$.

The fractional transversal # is the size of a minimum fract. transversal. $\tau^*(\mathcal{F})$

Obs: $\tau^*(\mathcal{F}) \leq \tau(\mathcal{F})$ But gap could be large

Exercise $\mathcal{F} = \binom{[n]}{n/2}$, $\tau(\mathcal{F}) = n/2 + 1$

But notice that this $\mathcal{F}$ has $\tau^*(\mathcal{F}) \leq 2$

large VC dim $(= n/2)$

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Thm [Haussler-Welzl ε-net thm]

$\forall$ hyp. $\mathcal{F}$ w/ VC dim d satisfies

$$\tau(\mathcal{F}) \leq 16 \cdot d \cdot \tau^*(\mathcal{F}) \cdot \log\left(d \cdot \tau^*(\mathcal{F})\right)$$

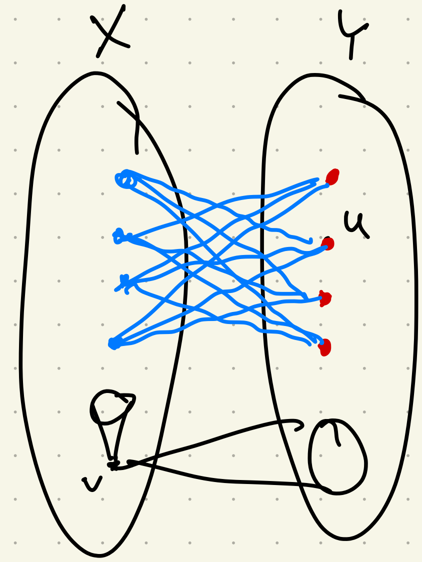
induced Q_3-free. $\delta(G) = \Omega(n)$

Pf (t-T) • Consider a max cut $X \cup Y$ of $V(G)$

so $\forall x \in X$, $d(x,Y) \geq d(x,X)$

$$\delta(G) \geq cn \Rightarrow d(x,Y) \geq \delta(G)/2 \geq cn/2$$

Let $\mathcal{F} = N_X = (Y, \{N(v) : v \in X\})$



Assigning $\varphi(u) = \frac{2}{cn}$, we get

that $\tau^*(\mathcal{F}) \leq 2/c$ (bdd!)

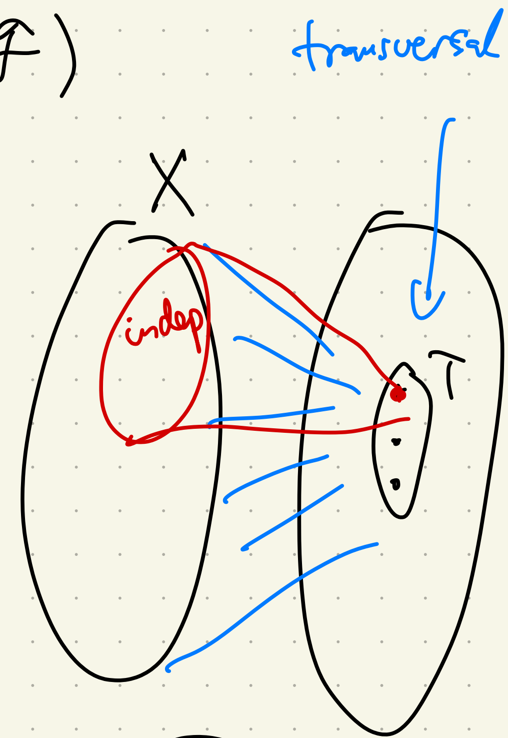
- Next, observe that VC dim of $\mathcal{F} \leq 3 = d$

  as G has no induced Q_3.

- ε-net thm $\Rightarrow$ $\tau(\mathcal{F}) \leq 16 \cdot d\, \tau^*(\mathcal{F}) \cdot \log(d\, \tau^*(\mathcal{F}))$

$$= O_c(1)$$

- Note that $\chi(G[X]) \leq \tau(\mathcal{F})$



A transversal T of size $\tau(\mathcal{F})$

$\Rightarrow$ $N(T) \supseteq X$ by defn.

of $\mathcal{F} = N_X$

Δ-freeness $\Rightarrow$ $N(t)$ indep. $\forall t \in T$

$\Rightarrow$ proper $|T|$-coloring of $G[X]$.