Lecture 10

Def: (Chr. threshold) The chromatic threshold of a graph H is

$$\delta_\chi(H) := \inf\{\alpha \in [0,1] : \exists\, C(H,\alpha) \text{ s.t. } \forall \text{ n-vx } H\text{-free graph } G \text{ with } \delta(G) \geq \alpha n \Rightarrow \chi(G) \leq C\}$$

That is, $\forall\, \alpha < \delta_\chi(H)$, the chr. # of n-vx H-free graph w/ mindeg αn could be arbitrarily large; while when $\alpha > \delta_\chi(H)$, then the chr. # is necessarily bounded.

$$\delta_\chi(K_3) = 1/3, \qquad \delta_\chi(K_r) = \frac{2r-5}{2r-3}.$$

- Thomassen: $\delta_\chi(C_{2k+1}) = 0, \quad \forall\, k \geq 2.$

That is, $\forall$ n-vx, C_{2k+1}-free graph G w/ $\delta(G) = \Omega(n)$

$$\chi(G) = O(1).$$

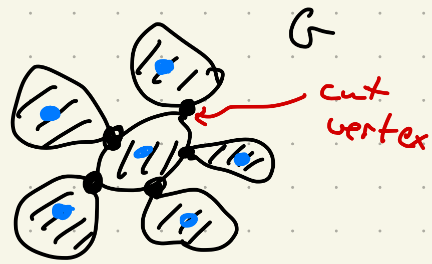
Thm (Thomassen 07) $\forall\, \varepsilon > 0$, $k \geq 2$, $\exists\, C(\varepsilon, k)$ s.t. TFH

Let G be an n-vx C_{2k+1}-free graph.

If $\delta(G) \geq \varepsilon n \Rightarrow$ then $\chi(G) \leq C$.

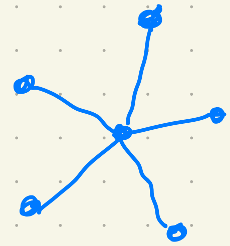
Def. • A block of a graph G is a maximal induced 2-connected subgraph

• The block decomposition of G



$\Downarrow$

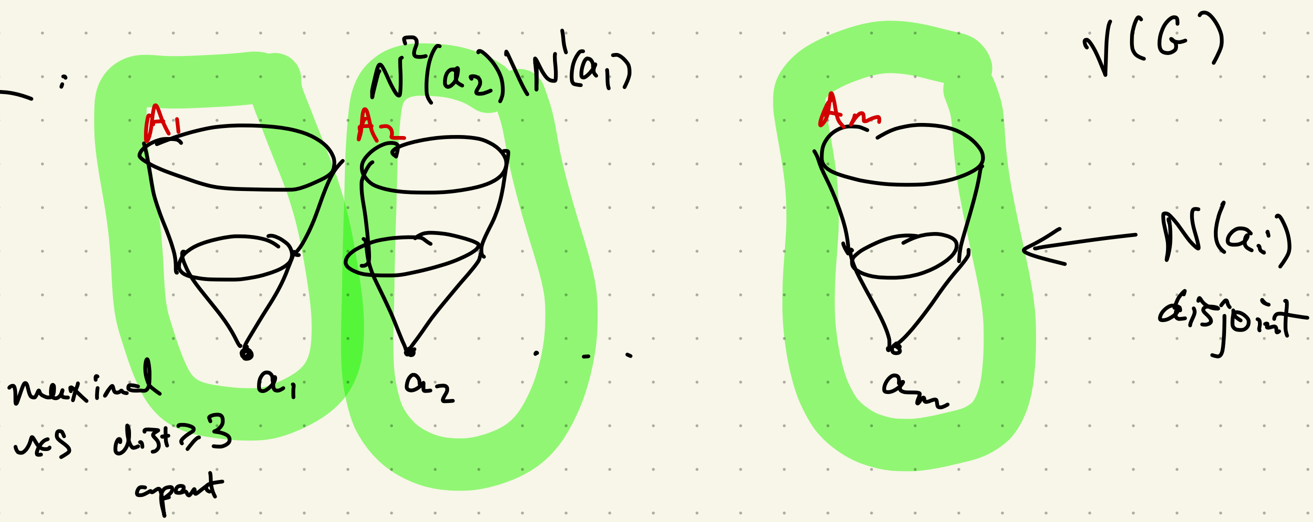
- The block decomp. auxiliary graph is acyclic, i.e. a forest.



Prop i) If G is nonbip. $\Rightarrow \exists$ a nonbip. block

Exercise ii) $\exists$ a block B s.t. $\chi(G) = \chi(B)$

Idea :



Pf
- Take a maximal set of vxs w/ pairwise distance ≥ 3, say $\{a_1, \dots, a_m\}$.
- As a_i's are of dist ≥ 3 apart $\Rightarrow$ $N(a_i)$, $i \in [m]$ are disjoint.

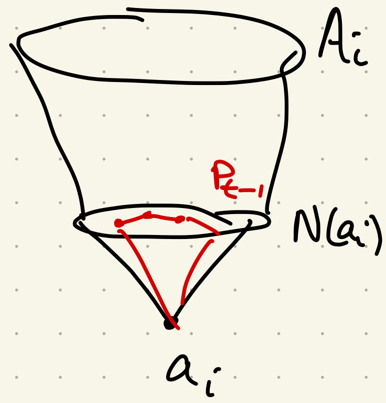
  $N(a_i) \geq \delta(G) \geq \varepsilon n \overset{\text{disjointness}}{\Longrightarrow} m \leq 1/\varepsilon$
- Let $A_i = N^2(a_i) \setminus \left(\bigcup_{j<i} N^2(a_j) \right)$

  Maximality $\Rightarrow$ $\{a_i\} \cup N(a_i) \cup A_i$, $i \in [m]$ form a partition of $V(G)$.

Thus, it suffices to show that each part $\{a_i\} \cup N(a_i) \cup A_i$ has bdd chr. #.

- Let $t = 2k+1$, G is C_t-free

- $N(a_i)$ is $(t-3)$-degenerate, hence $(t-2)$-colorable, for o.w. $N(a_i)$ contains a P_{t-1}, which



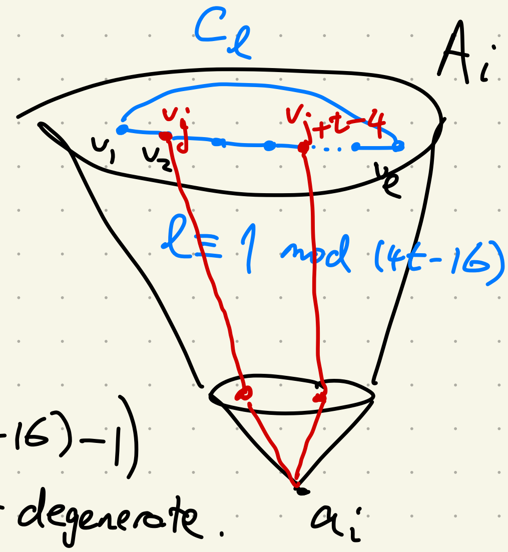
$\left(\delta(G) \geq a-1 \Rightarrow P_a, \quad \text{no } P_{t-1} \Rightarrow \delta(G) \leq t-3 \right)$

together w/ a_i form a copy of C_t ↯.

Thm (✱) (Thomassen 83) $\forall p \in \mathbb{N}$ $\exists D(p)$ s.t. every 2-connected, nonbipartite graph G w/ $\delta(G) \geq D(p)$ contains cycles of all length modulo p.

$t = 2k+1 \geq 5$

<u>Claim</u> $G[A_i]$ is $D(4t-16)$-colorable.

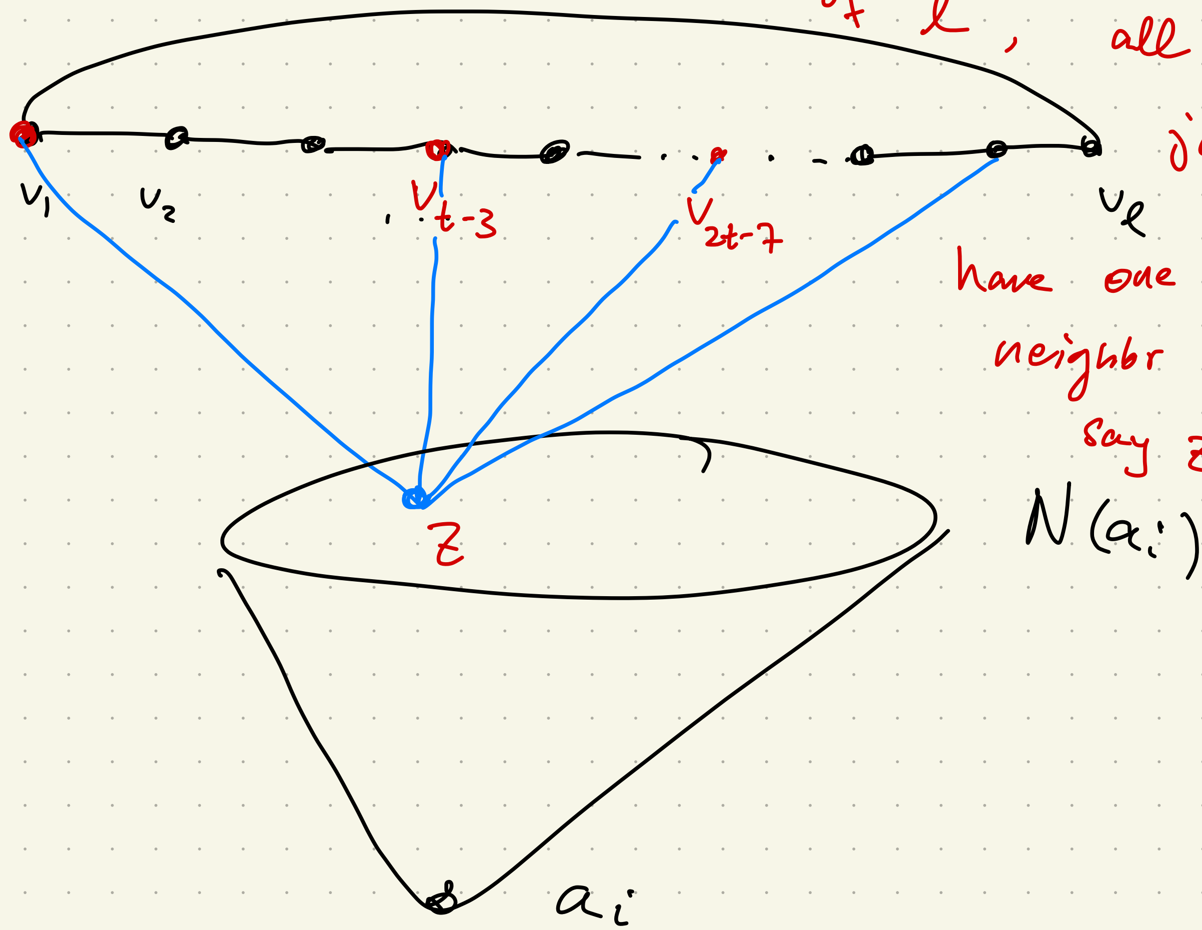


<u>Pf</u>: • Consider a block B w/ $\chi(B) = \chi(G[A_i])$.

We need to show that B is $(D(4t-16)-1)$-degenerate.

- Supp. not, Thm(✱) $\Rightarrow$ B contains a cycle of length $1 \mod (4t-16)$, say $v_1, v_2, \ldots, v_\ell$, $\ell \equiv 1 \mod (4t-16)$

• Now $\forall j \in [\ell]$, v_j and v_{j+t-4} cannot have different neighbrs in $N(a_i)$ for o.w. we get a copy of C_t ↯. $\Longrightarrow$ Therefore, by the choice of ℓ, all v_j's $j \in [\ell]$ have one common neighbr in $N(a_i)$ say z.



Take a sub path of order $t-1$ $(\leq 4t-16)$ we get a C_t (w./ z) ↯. ▨

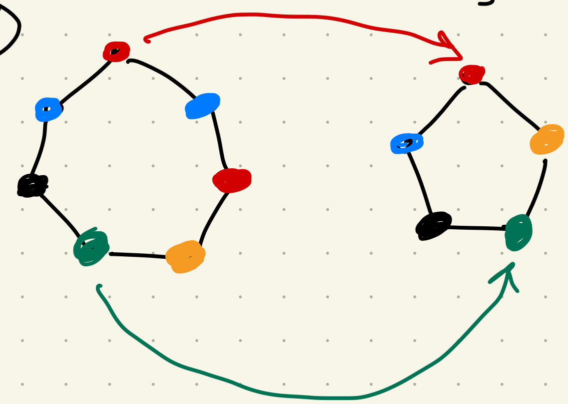
Thus $\chi(G) \leq m \cdot \big(\underbrace{(t-2)}_{N(a_i)} + \underbrace{D(4t-16)}_{A_i} \big)$

$\leq \frac{1}{\varepsilon} (\ldots) = O_{\varepsilon,k}(1)$ ▨

§ Homomorphism threshold.

__Def__: A homomorphism φ from G to Γ is a map $\varphi : V(G) \longrightarrow V(\Gamma)$ that preserves the adjacencies. That is, if $uv \in E(G)$ then $\varphi(u)\varphi(v) \in E(\Gamma)$.

Ex $C_7 \xrightarrow{\varphi} C_5$



__Obs__: $\exists\ G \xrightarrow[\varphi]{\text{homomorphism}} \Gamma$

$\Longleftrightarrow \quad G \subseteq \Gamma(t)$

- Notice that $\chi(G) \leq C$ is the same as G admits a homomorphism into K_C

$$\left(G \xrightarrow{\text{hom.}} K_C \right)$$

One may further require the homomorphic image to be H-free and has bdd order.

<u>Def</u>: (Homomorphism threshold) The homomorphism threshold of a graph H is defined as

$$\delta_{hom}(H) = \inf \{ \alpha \in [0,1] : \exists \text{ an } H\text{-free graph } \Gamma = \Gamma(H,\alpha) \text{ s.t. every } H\text{-free graph } G \text{ w/ } \delta(G) \geq \alpha|G| \text{ satisfies } G \xrightarrow{hom.} \Gamma \}$$

- Since $G \xrightarrow{hom} \Gamma \implies \chi(G) \leq |\Gamma| = O(1)$.

$$\implies \delta_{hom}(H) \geq \delta_\chi(H)$$

We shall see that

$$\delta_{hom}(K_r) = \delta_\chi(K_r) = \frac{2r-5}{2r-3}$$

- Łuczak 06: $\delta_{hom}(K_3) = 1/3$.
- Goddard-Lyle: K_r.

Łuczak's pf uses Szemerédi's regularity and so the size of the homomorphic image Γ is a tower-type funct. of α.

We shall see a recent probabilistic pf of [Oberkampf – Schacht. 2020], which gives only a double exponential bd.

<u>Thm</u> [O-S] For every $r \geq 3$ and every $\varepsilon > 0$, $\exists\ L = 2^{2^{\mathrm{poly}(r, 1/\varepsilon)}}$ s.t. for every n-vx K_r-free graph G w/ $\delta(G) \geq \left(\frac{2r-5}{2r-3} + \varepsilon\right) n$, there exists a K_r-free graph Γ on $\leq L$ vxs w/ $G \xrightarrow{\text{hom.}} \Gamma$.

<u>Δ-case</u>: $\forall$ n-vx Δ-free G w/ $\delta(G) \geq (\frac{1}{3} + \varepsilon) n$ $\exists\ |\Gamma| \leq 2^{2^{\mathrm{poly}(1/\varepsilon)}}$ s.t. $G \xrightarrow{\text{hom}} \Gamma$.
Γ Δ-free

- Can the $\delta_{\text{hom}}(H) \geq \delta_{\chi}(H)$ be strict? ineq.

hom. threshold · Ebsen – Schacht 17
odd cycle
· Letzter - Snyder 19

$$\left(0 = \delta_\chi(C_{2k+1}) \leq \right) \delta_{hom}(C_{2k+1}) \leq \frac{1}{2k+1}$$

$$\delta_{hom}(\mathcal{C}_{2k+1}) = \frac{1}{2k+1}$$

$\mathcal{C}_{2k+1} = \{C_3, \ldots, C_{2k+1}\}$

- Sankar (22+): $\delta_{hom}(C_{2k+1}) > 0$