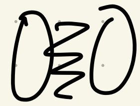
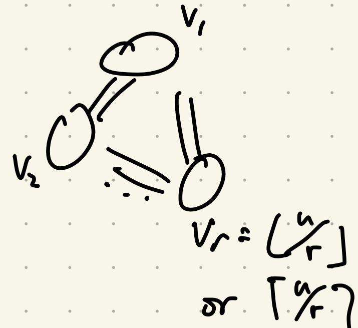
Lecture 2

Recall

- $ex(n,H) = \max\{e(G) : |V(G)|=n,\ H\text{-free}\}$
- Mantel's thm: $ex(n,K_3) = \lfloor \frac{n^2}{4} \rfloor$



- Turán's thm: $ex(n, K_{r+1}) = e(T_{n,r})$, $r \geq 2$



- Caro-Wei: $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$

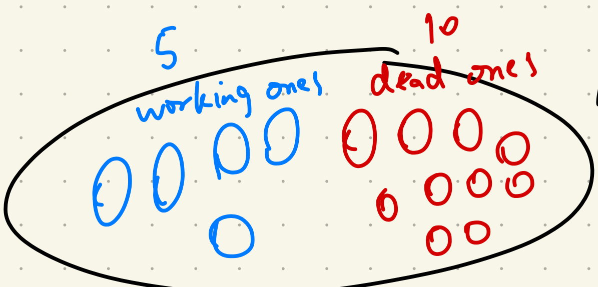
Ex: Find an example showing tightness of Caro-Wei

- About Erdős-Sós's conj:

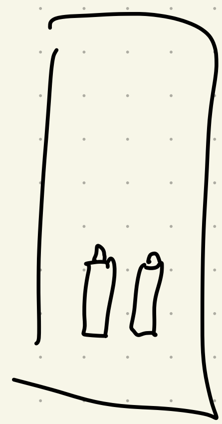
Known: Erdős-Gallai 59: $ex(n, P_k) \leq \frac{(k-2)}{2} \cdot n$ ← k-vertex path

§ 1.2.1 Quick applications of Turán's thm.

1) Ex (Nika Salia): Remote TV control needs two working batteries.



bag of 15 batteries

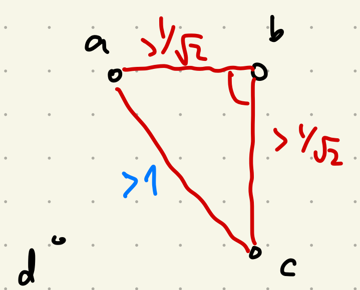


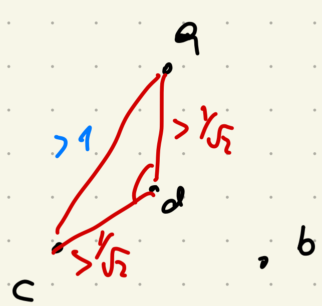
How many times we have to try to guarantee to find a working pair of batteries?

Thm (Erdős) Let P be an arrangement of n pts in the plane w./ diameter ≤ 1. Then the number of pairs of pts w./ distance greater than $1/\sqrt{2}$ is at most $n^2/3$.

Pf:
- Build an auxiliary graph G w./ vertex set P and two pts $x, y \in P$ are adj. in G if and only if
$$d(x,y) > 1/\sqrt{2}$$
- It suffices to show that G is K_4-free, then Turán's thm $\Rightarrow e(G) \leq n^2/3$.

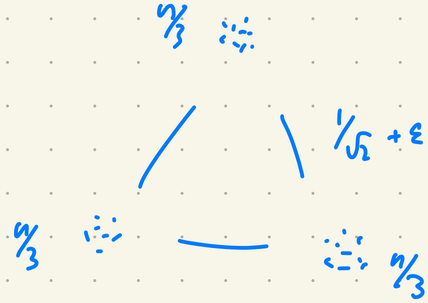
Say there is a K_4 on pts $a, b, c, d \in P$.
Only two kinds of arrangements of 4 pts in the plane:

1)


2)


In either case, $\exists$ a triangle w./ largest angle $\geq \pi/2$.
but then the corresponding the edge has length > 1
$\lightning$ $\operatorname{diam}(P) \leq 1$. $\blacksquare$

Rmk $n^2/3$ tight



§ 1.3 Symmetrization

§ 1.3.1 Zykov's Symm.

Zykov's symm. is a process in which we alter a graph G, one vertex at a time

- without decreasing the # of edges, and
- without increasing the clique #, denoted by $\omega(G)$.

the size of largest clique in G.

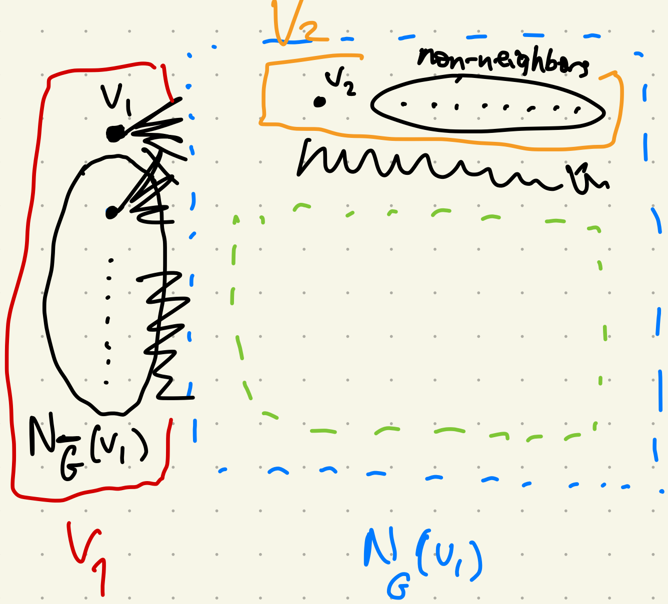
At the end of this process, we end up w/ a complete partite graph, which is a much simpler structure to deal with.

Pf: (Turán's thm via Symm.)

- Let G be an n-vertex K_{r+1}-free graph w/ maximum # of edges.

Pick a vertex, say $v_1 \in V(G)$, w/ maximum degree and symmetrizes all of its non-neighbors to v_1.

That is, $\forall u$ not adjacent to v_1 in G, set $N(u) := N(v_1)$ and let G_1 be the resulting graph.



G_1.

- G_1 is still K_{r+1}-free
- $e(G_1) \geq e(G)$ as v_1 has max. deg

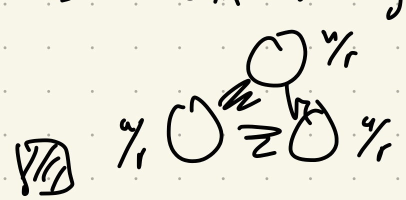
  $\Rightarrow e(G_1) = e(G)$ by maximality of G.
- $V_1 = V(G) \setminus N_G(v_1)$ is an indep set in G_1 and V_1 is completely joined to $V(G) \setminus V_1$ in G_1.

- We repeat this operation as follows. Pick $v_2 \in G_1[V(G) \setminus V_1]$ w/ max deg and symm. all its non-neighbors in $G_1[V(G) \setminus V_1]$ to v_2 and let G_2 be the resulting graph. Let $V_2 = V(G) \setminus (V_1 \cup N_{G_1}(v_2))$

  At the end, non-adjacency defines an equivalence relation $\Rightarrow$ the final graph is complete partite G^*

- As G^* is K_{r+1}-free and it has max. # edges

  $\Rightarrow G^*$ is balanced r-partite, i.e. $T_{n,r}$.

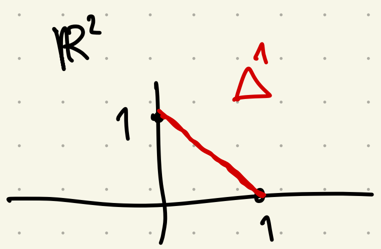


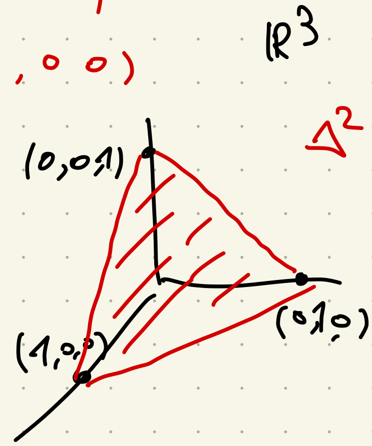
§1.3.1 Motzkin-Straus Symm.

<u>Def</u>: The $(n-1)$-dim simplex, denoted by Δ^{n-1}, is defined as

$$\Delta^{n-1} = \left\{ x \in \mathbb{R}^n : \forall\, i \in [n],\ x_i \geq 0 \text{ and } \sum_{i \in [n]} x_i = 1 \right\}$$

$$= \operatorname{conv}\{ \underset{(0,0,\dots,\overset{i}{1},0,0)}{e_i},\ i \in [n] \}$$





- Given a graph G, we write A_G for its adjacency matrix.

$$A_G = \begin{array}{c} \\ i \\ \\ \end{array} \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}, \quad \begin{array}{c} \vdots \\ j \end{array}$$

if $ij \in E(G)$

- $$x^T A_G x = \sum_{ij \in [n] :\, ij \in E(G)} x_i x_j$$

- We can think of $x \in \Delta^{n-1}$ as a weight funct. on $V(G)$, then x naturally corresponds a (vertex) weighted graph G_x, and

$$x^T A_G x = 2\, e(G_x) = 2 \sum_{ij \in [n] :\, ij \in E(G)} x_i x_j$$

Thm (Motzkin - Straus)

Let G be an n-vertex graph w./ clique # $\omega(G) = r$ and $x \in \Delta^{n-1}$.

$\Rightarrow$ Then $\exists\, y \in \Delta^{n-1}$ s.t.

- $y^T A_G y \geq x^T A_G x$
- $\operatorname{supp}(y)$ corresponds to a clique in G.

In particular, $x^T A_G x \leq \dfrac{r-1}{r}$, $\forall\, x \in \Delta^{n-1}$.

Rmk: Let $x = (1/n, 1/n, \dots, 1/n) \in \Delta^{n-1}$

$$\Rightarrow \quad x^T A_G x = \frac{2}{n^2} e(G) \underset{K_{r+1}\text{-free}}{\underset{\Uparrow}{\leq}} \frac{r-1}{r}$$

We recover Turán's thm.

The idea of M-S is "mass transportation": if x has mass on two coord. corresponding to a pair of non-adj. vertices, then we can move the mass from one coord. to the other w./ decreasing $x^T A_G x$. This eventually leads to a vector whose supp. induces a clique.

<u>Pf</u>: Write $f(z) = z^T A_G z$.

- Take $y \in \Delta^{n-1}$ w/ minimal support s.t

$$f(y) \geq f(x).$$

- Suppose to the contrary that $\mathrm{supp}(y)$ contains a pair of non-adj. vertices, say coord. 1 & 2.

Note that $\forall\ z = (z_1, -z_1, 0, \cdots, 0)^T$,

we have $z^T A_G z = 0$ as $\{1,2\} \notin E(G)$

Consequently, $\forall\ \alpha \in \mathbb{R}$, writing $\mathbf{a} := 2\alpha A_G \cdot y$

$$\Rightarrow f(y+\alpha \cdot z) = (y+\alpha z)^T A_G (y+\alpha z)$$

$$= f(y) + 2\alpha z^T A_G y$$

$$= f(y) + a^T \cdot z.$$

- $a^T \cdot z = (a_1 - a_2) z_1 \geq 0$

choose appropriate $z_1 \Rightarrow f(y+\alpha \cdot z) \geq f(y)$

- $y' = y + \alpha \cdot z = (y_1 + \alpha z_1, y_2 - \alpha z_1, y_3, \cdots, y_n)^T$

sum of coord still $= 1$

We can choose appropriate α so that one of $y_1 + \alpha z_1$ and $y_2 - \alpha z_1$ becomes 0 while the other one is still positive

$\Rightarrow$ $\mathrm{supp}(y') \subsetneq \mathrm{supp}(y)$ while $f(y') \geq f(y)$

$\lightning$ minimality of y.

Thus $\mathrm{supp}(y)$ induces a clique.

We leave it as an exer to show the "In particular" part. $\blacksquare$