Lecture 1

Problems in extremal combinatorics can often be phrased as follows:

Given : • $\mathcal{C}$: a finite fam. of discrete objects (w./ certain properties)

• f : parameter defined over members of $\mathcal{C}$.

Problem : $\max\limits_{G \in \mathcal{C}}$ (or min) $f(G)$

§1 Turán type problem

Puzzle: we can choose n irrational numbers, $x_1, \dots, x_n$.

Goal : max # pair (x_i, x_j) s.t. $x_i + x_j$ rational.

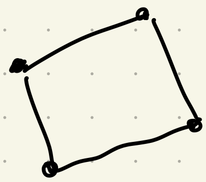
Problem (Turán type) How dense a graph can be without containing a given graph as a subgraph?

Formally, given a graph H, we say a graph G contains a copy of H, denoted by $H \subseteq G$, if there is an injective map $\varphi : V(H) \to V(G)$ preserving the adjacencies, i.e. $\forall\, uv \in E(H) \Rightarrow \varphi(u)\varphi(v) \in E(G)$.

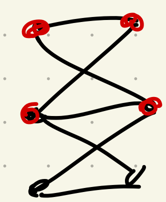
When G does NOT contain H as a subgraph, we say that G is **H-free**.

<u>Ex</u>

H

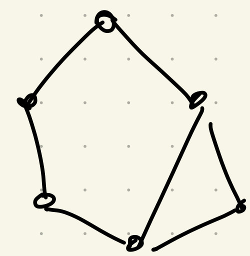


G_1



$H \subseteq G_1$

G_2



H-free

$H \not\subseteq G_2$

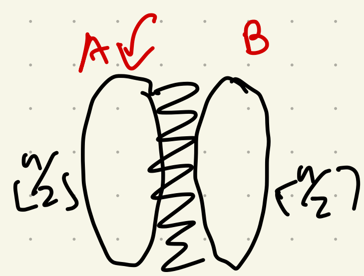
<u>Def</u>: The extremal number (Turán number) of H, denoted by **$ex(n, H)$**, is the max. # edges in an n-vertex H-free graph.

An extremal graph for H is an n-vertex graph w/ $ex(n, H)$ edges.

§ 1.1 Mantel's thm

<u>Thm</u> (Mantel 1907) n-vx G Δ-free $\Rightarrow$

$$e(G) \le ex(n, K_3) = \lfloor n^2/4 \rfloor$$

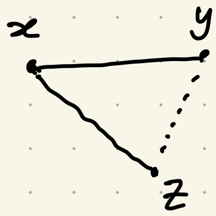


<u>Ex</u> Solve the rational pair puzzle using Mantel's thm.

Pf 1 · $\forall\ xy \in E(G)$ & $\forall z \in V(G)$

z cannot be adj. to both x & y

by Δ-freeness



$$\Rightarrow \quad \boxed{d(x) + d(y) \le n \qquad (*)}$$

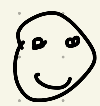
· Summing $(*)$ over all edges, we get

$$\sum_{xy \in E(G)} \big(d(x) + d(y)\big) \le n \cdot e(G) \qquad \cdots (1)$$

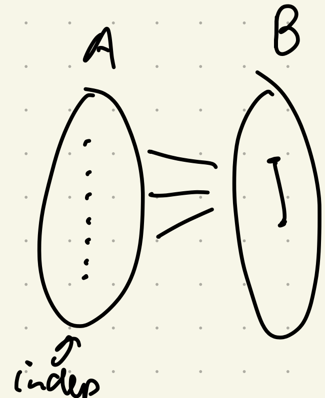
· Expanding the LHS, each vertex x in G contributes exactly $d(x)^2$ to this sum.

$$\Rightarrow \sum_{xy \in E(G)} \big(d(x) + d(y)\big) = \sum_{x \in V(G)} d(x)^2$$

$$\overset{C\text{-}S}{\geq} \frac{1}{n} \Big(\sum_{x \in V(G)} d(x) \Big)^2 = \frac{1}{n} \cdot 4 e(G)^2 \quad \cdots (2)$$

Combining (1) and (2) $\Rightarrow$  $\blacksquare$

Pf 2 · Let A be a maximum indep. set in G and $B = V(G) \setminus A$



· Δ-free $\Rightarrow$ $\forall v \in V(G)$ $N_G(v)$ is an indep set

$$\Rightarrow \quad d(v) \leq |A|$$

. As A is an indep set, every edge of G is incident to a vertex in B

$$\Rightarrow e(G) \leq \sum_{v \in B} d(v) \leq |B| \cdot |A| \overset{\text{AM-GM}}{\leq} \left(\frac{|A|+|B|}{2} \right)^2 = \frac{n^2}{4}$$

$\blacksquare$

Ex. Prove that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the UNIQUE extremal graph for Δ.

Ex. $\forall$ graph G w/ average degree $d(G)$ at least d contains a subgraph H w/ min deg $\delta(H) \geq d/2$.

Ex. For any k-vertex tree T,

$$ex(n, T) < (k-1) \cdot n$$

Improving this upp. bd by a factor of 2 is the well-known Erdős-Sós conj.

Erdős-Sós conj Let G be an n-vertex graph w/ $e(G) \geq \frac{(k-2) \cdot n}{2} + 1 \Rightarrow$ it contains every k-vertex tree.

§ 1.2 Turán's thm.

<u>Def</u>: • A proper k-coloring of a graph G is a map $c: V(G) \to [k]$ s.t. $\forall$ edge $xy \in E(G)$

$$c(x) \neq c(y)$$

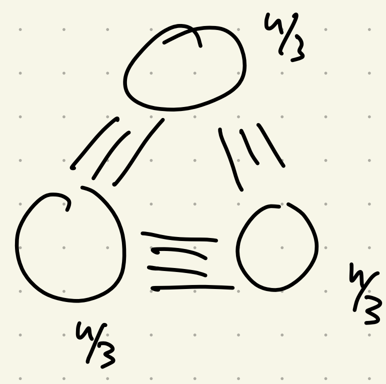
• The chromatic number of G, denoted by $\chi(G)$, is the min # colors needed to properly color G.

• We say that G is r-partite if $\chi(G) \le r$.

<u>Turán graph</u> Let $r \in \mathbb{N}$, the r-partite Turán graph on n vertices, denoted by $T_{n,r}$, is the balanced complete r-partite n-vertex graph. i.e. each partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Clearly, $T_{n,r}$ is K_{r+1}-free.

$r = 3$



<u>Turán thm</u> (1941) $\forall$ n-vx G

if G is K_{r+1}-free $\Rightarrow e(G) \le e(T_{n,r})$,

i.e. $\quad ex(n, K_{r+1}) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - O(r)$

Furthermore, $T_{n,r}$ is the unique extremal graph.

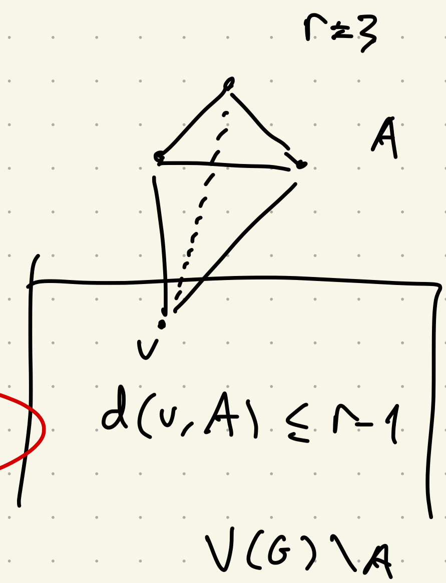
Pf 1 • Induction on the number of vertices n.

Base: Trivial when $n \le r$.

Inductive: Let G be an n-vx extremal graph for K_{r+1}.

. Maximality of G $\Rightarrow$ G contains a copy of K_r, say A.

. $\forall\ v \in V(G) \setminus A := B$ has $\le r-1$ neighbors in A, i.e. $d(v, A) \le r-1$.



$$\Rightarrow e(G) \le e_G(A) + e_G(A,B) + e_G(B)$$

I.H.

$$\le \binom{r}{2} + |B|\cdot(r-1) + s^{\ell}\cdot(s-1)^{r-\ell}$$

$$\cdots \le (s+1)^{\ell}\cdot s^{r-\ell}$$

$$\frac{n = s\cdot r + \ell}{0 \le \ell \le r-1}$$

$|B| = n - r = (s-1)r + \ell$

r-parts $\begin{cases} \ell \text{ parts} & s \\ (r-\ell) \text{ parts} & s-1 \end{cases}$

∎

Def: The indep. number of a graph G, denoted by $\alpha(G)$, is the max. size of an indep. set (a set inducing no edges)

Thm (Caro-Wei) $\forall\ G$,

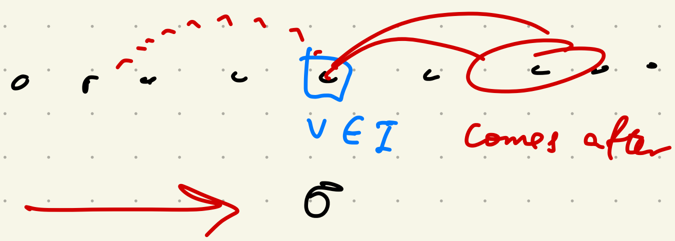
$$\Rightarrow \alpha(G) \ge \sum_{v \in V(G)} \frac{1}{d(v)+1}.$$

Ex: Derive Turán's thm from Caro-Wei.

Pf: (1^{st} moment). Take a uniform random ordering ($[n] = V(G)$) of $V(G)$, say $\sigma \sim S_n$, and let I be the set of all vertices v with no neighbors preceding itself in this ordering i.e.

$$I = \{ v \in V(G) : \forall u \in N(v),\ \sigma(u) > \sigma(v) \}$$

• By choice, I is an indep set.



$$\mathbb{E}(|I|) \overset{\text{linearity of expectation}}{=} \sum_{v \in V(G)} \Pr\left(\forall u \in N(v),\ \sigma(u) > \sigma(v) \right)$$

$$= \sum_{v \in V(G)} \frac{1}{d(v)+1}.$$

Thus, there exists an indep set of size $\geq \sum \frac{1}{\cdots}$

$$\Rightarrow \alpha(G) \geq \sum \frac{1}{\cdots}$$