Topological Combinatorics Part 2 -2

Review

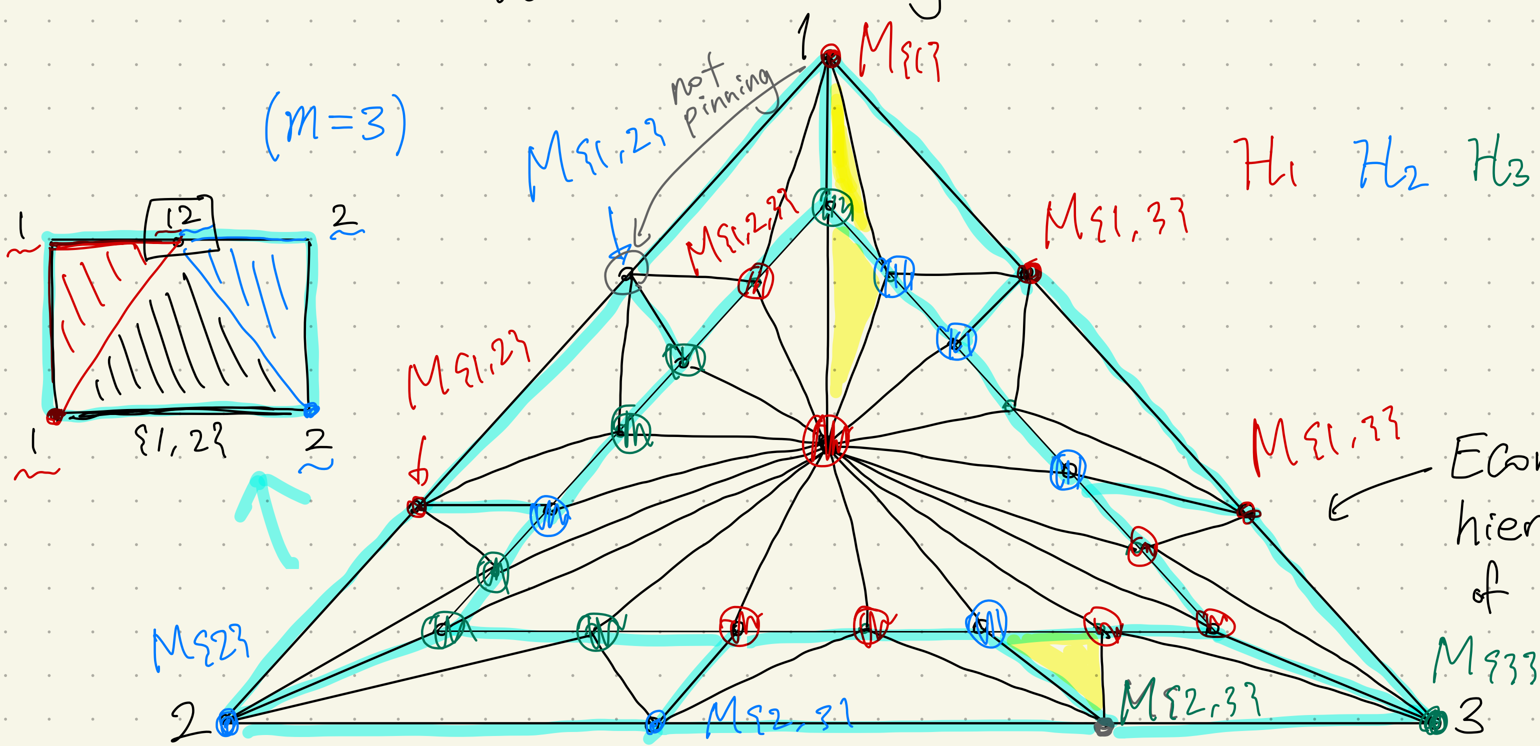
Hypergraph Hall Theorem (Aharoni–Haxell, 2000)

$\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_m$: hypergraphs on V, $\mathcal{H}_I = \bigcup_{i \in I} \mathcal{H}_i$.

$\forall I \subseteq [m]$ $(\neq \emptyset)$, $\exists M_I$: a matching in $\mathcal{H}_I$ that cannot be pinned by $< |I| - d$ edges of $\mathcal{H}_I$

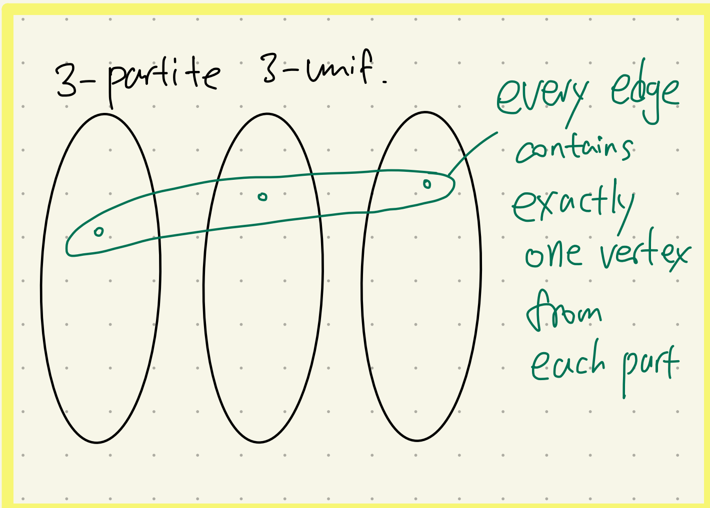
$\Rightarrow \exists$ rainbow matching of size $m - d$

$(m=3)$



Economically hierarchic triangulation of 2-dim. simplex

<u>THM</u> (Aharoni, 2001) $\mathcal{H}$: 3-partite 3-uniform hypergraph.



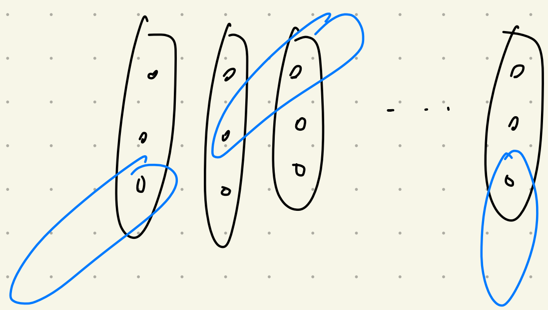
$$\Rightarrow \left(\begin{array}{c}\text{min. \# vertices}\\ \text{to meet all edges}\end{array}\right) \leq 2\left(\begin{array}{c}\text{maximal size}\\ \text{of a matching}\end{array}\right)$$

$$\underbrace{\tau(\mathcal{H})}_{\text{transversal}} \leq 2\nu(\mathcal{H})$$

($r=3$ case of Ryser's conjecture).

Trivial bound : $\tau(\mathcal{H}) \leq 3\nu(\mathcal{H})$. (for any 3-unif. hypergraph)

($\because$) Take a matching of size $\nu(\mathcal{H})$



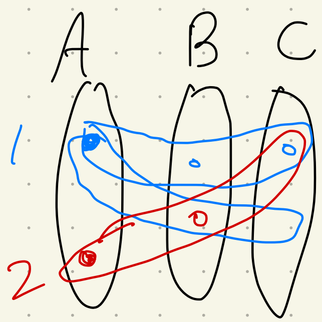
M : maximal matching

$\Downarrow$

any other edge in $\mathcal{H}$ should meet at least one edge of M.

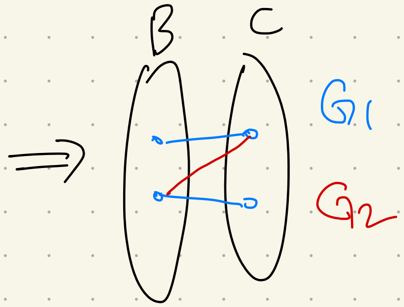
$\Rightarrow V(M)$ meets all edges.

proof) $\mathcal{H}$: 3-partite 3-unif. hypergraph on $A \cup B \cup C$



By considering A as a color class,

$\mathcal{H}$ induces bipartite graphs $G_1, G_2, \dots, G_{|A|}$ on $B \cup C$.

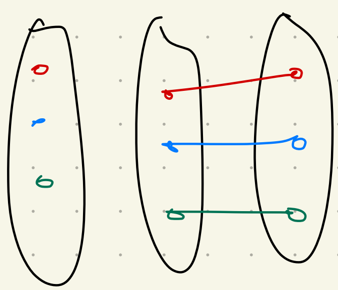


$d :=$ min. positive integer such that

$\forall I \subseteq [|A|]$ $(\neq \phi)$ $\exists M_I$ in $\bigcup_{i \in I} G_i$

that cannot be pinned by

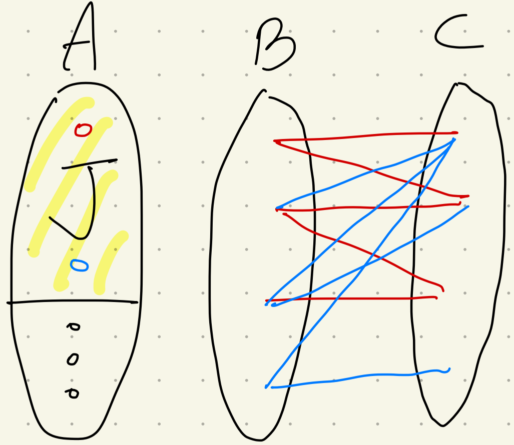
$< |I| - d$ edges of $\bigcup_{i \in I} G_i$.



① $\Rightarrow \exists$ rainbow matching of size $|A| - d$.

$\Rightarrow \nu(H) \geq |A| - d$.

② Let $J \subset [|A|]$ s.t. $\forall$ matching in $\bigcup_{i \in J} G_i$, it can be pinned by $|J| - d$ edges.

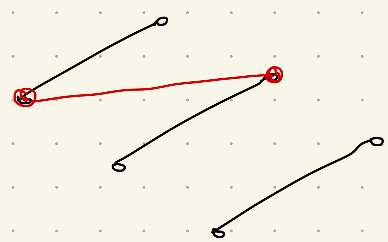


$(A\setminus J) \cup \left[\text{Transversal of } \bigcup_{i\in J} G_i\right] \Rightarrow$ gives a transversal of $\mathcal{H}$

$$\tau\left(\bigcup_{i\in J} G_i\right) \overset{\text{König}}{=} \nu\left(\bigcup_{i\in J} G_i\right) \le 2(|J|-d)$$

If $\nu\left(\bigcup_{i\in J} G_i\right) > 2(|J|-d)$.

then $\exists$ matching M of size $> 2(|J|-d)$



An edge can pin at most two edges

Since an edge can pin ≤ 2 edges from M,

M cannot be pinned by $|J|-d$ edges, contradiction

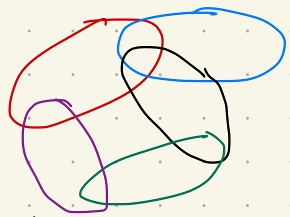
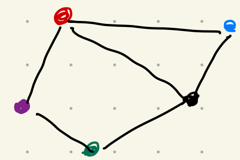
① $\nu(\mathcal{H}) \ge |A| - d$

② $\tau(\mathcal{H}) \le (|A|-|J|) + 2(|J|-d)$

$= |A| + |J| - 2d \le 2|A| - 2d \le 2\nu$

Ryser's Conjecture $\forall r \geq 2$, $\mathcal{H}$: r-partite r-unif hypergraph

$$\tau(\mathcal{H}) \leq (r-1)\,\nu(\mathcal{H}).$$ (open from $r=4$)

$\mathcal{H}$: hypergraph  $\Longleftrightarrow$  G : intersection graph

$w(\mathcal{H})$ width : min k s.t. $\mathcal{H}$ can be pinned by k edges

$mw(\mathcal{H})$ matching width : min k s.t. any matching of $\mathcal{H}$ can be pinned by k edges

$w(\mathcal{H})$ / $mw(\mathcal{H})$	$\gamma(G)$ / $i\gamma(G)$
$\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_m$	$G_1, G_2, \cdots, G_m$
$\forall I$, $mw(\mathcal{H}_I) \geq \lvert I \rvert$	$\forall I$, $i\gamma(G_I) \geq \lvert I \rvert$
$\Rightarrow \exists$ rainbow matching	$\Rightarrow \exists$ rainbow independent set (independent transversal)

$\gamma(G)$ domination number : minimal size of $S \subseteq V(G)$ s.t. $V(G) = S \cup N(S)$

$i\gamma(G)$ independence domination number : min. k s.t. $\forall A \subseteq V(G)$: independent in G

$\exists S \subseteq V(G)$ s.t. $A = S \cup N(S)$

THM (Meshulam, 2001)

$\mathcal{H}_1, \cdots, \mathcal{H}_m$ hypergraphs.

$\forall \underset{\neq\emptyset}{I} \subseteq [m]$, $\quad w(\mathcal{H}_I) > 2(|I|-1)$ or $mw(\mathcal{H}_I) > |I|-1$

$\Rightarrow \exists$ rainbow matching.

K : (abstract) simplicial complex

$\eta_H(K)$ homology connectivity : max k s.t. $\tilde{H}_i(K) = 0 \quad \forall\, i \leq k-2$ why -2?

$\boxed{\text{OBS} \quad \text{join} \quad K * L = \{\sigma \cup \tau : \sigma \in K, \tau \in L\} \Rightarrow \eta_H(K*L) = \eta_H(K) + \eta_H(L)}$

THM (Topological Hall theorem)

K : simplicial complex on V, $\quad V = V_1 \cup V_2 \cup \cdots \cup V_m$ partition

If $\forall I \subseteq [m]$ $(\neq \emptyset)$, $\eta_H(K[\bigcup_{i\in I} V_i]) \geq |I|$, then $\exists$ rainbow simplex of K

$\{x_1, \cdots, x_m\} \in K$ s.t. $x_i \in V_i$.

H hypergraph $\Longrightarrow$ G intersection graph $\Longrightarrow$ $\mathcal{I}(G)$ independence complex

$$\mathcal{I}(G) = \{A \subseteq V(G) : A \text{ is independent}\}$$

$(w(H) > 2(k-1)$ or $mw(H) > k-1)$

Prop If $\gamma(G) > 2(k-1)$ or $i\gamma(G) > k-1$

then $\eta_H(\mathcal{I}(G)) \geq k$.

Homotopy nerve theorem

$\mathcal{F} = \{K_1, K_2, \cdots, K_m\}$, every non-empty intersection is contractible

$\Longrightarrow$ $N(\mathcal{F}) \underset{\text{homotopy equiv.}}{\simeq} \bigcup_{i\in[m]} K_i$

Homology nerve theorem

$\mathcal{F} = \{K_1, K_2, \cdots, K_m\}$.

If $\forall k < m$, $\eta_H(\bigcap_{i\in I} X_i) \geq k - |I| + 2$

for all I with $|I| \leq k+1$,

then (i) $\tilde{H}_j(\bigcup_{i\in[m]} K_i) \cong \tilde{H}_j(N(\mathcal{F}))$ $\forall\, 0 \leq j \leq k$

(ii) $\tilde{H}_{k+1}(\bigcup_{i\in[m]} K_i) = 0 \Longrightarrow \tilde{H}_{k+1}(N(\mathcal{F})) = 0$.

proof of topological Hall by homology nerve thm

K : complex on $V_1 \cup V_2 \cup \cdots \cup V_m$, $V_I = \bigcup_{i \in I} V_i$

$\forall I \subseteq [m]$ $(\neq \emptyset)$, $\eta_H(K[V_I]) \geq |I|$

Assume there is no rainbow simplex $\Longrightarrow$ $K = \bigcup_{j \in [m]} Y_j$

$Y_j = K[V_{[m] \setminus \{j\}}] \Rightarrow \{Y_1, Y_2, \ldots, Y_m\} =: \mathcal{F}$

$N(\mathcal{F}) \cong \partial 2^{[m]} \cong (m-2)$-dim. sphere $\Rightarrow \tilde{H}_i(N(\mathcal{F})) \neq 0$ iff $i = m-2$.

$\bigcap_{j \in J} Y_j = K[V_{[m] \setminus J}] \neq \emptyset$. $(J \neq [m])$	$\bigcap_{j \in [m]} Y_j = \emptyset$

$\eta_H(\bigcap_{j \in J} Y_j) \geq m - |J|$ $\Rightarrow$ Set $k = m-2$ in homology nerve thm
$J \neq [m]$

contradiction

$\Rightarrow \tilde{H}_{m-2}(N(\mathcal{F})) \cong \tilde{H}_{m-2}(K)$

$\tilde{H}_{m-2}(N(\mathcal{F})) \neq 0$, $\tilde{H}_{m-2}(K) = 0$ $(\eta(K[V_{[m]}]) \geq m)$