8/2 (Tue)

- Brouwer fixed-point theorem
- Sperner's lemma
- KKM theorem (Knaster – Kuratowski – Mazukiewicz)

$$\begin{tikzcd}
 & \text{Brouwer (1912)} \arrow[dl, "\substack{1974 \\ (\text{Yaseloff})}"'] & \\
\text{Sperner (1928)} \arrow[rr, "1929"'] & & \text{KKM (1929)} \arrow[ul, "1929"']
\end{tikzcd}$$

Algebraic topology	Combinatorics	Set covering	
Brouwer fixed-point theorem	Sperner's lemma	KKM theorem	– equivalent
Borsuk – Ulam theorem	Tucker's lemma	Lusternik – Schnirelmann theorem	– equivalent

Borsuk – Ulam $\Rightarrow$ Brouwer fixed-point thm

Brouwer fixed-point theorem

: $f: B^d \to B^d$ continuous map $\Rightarrow$ $x_0 \in B^d$ s.t. $f(x_0) = x_0$.

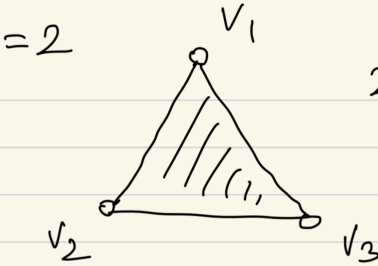
$B^d := \{ x \in \mathbb{R}^d : \|x\| \leq 1 \}$

Sperner's lemma

Δ^n : n-simplex on verticies $\underbrace{v_1, \cdots, v_{n+1}}_{\text{ground set}}$

$:= 2^{\{v_1, \cdots, v_{n+1}\}}$

ex $n=2$



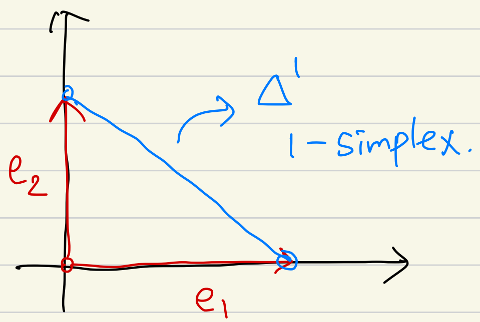
2-simplex Δ^2

Typical geometric realization of Δ^n

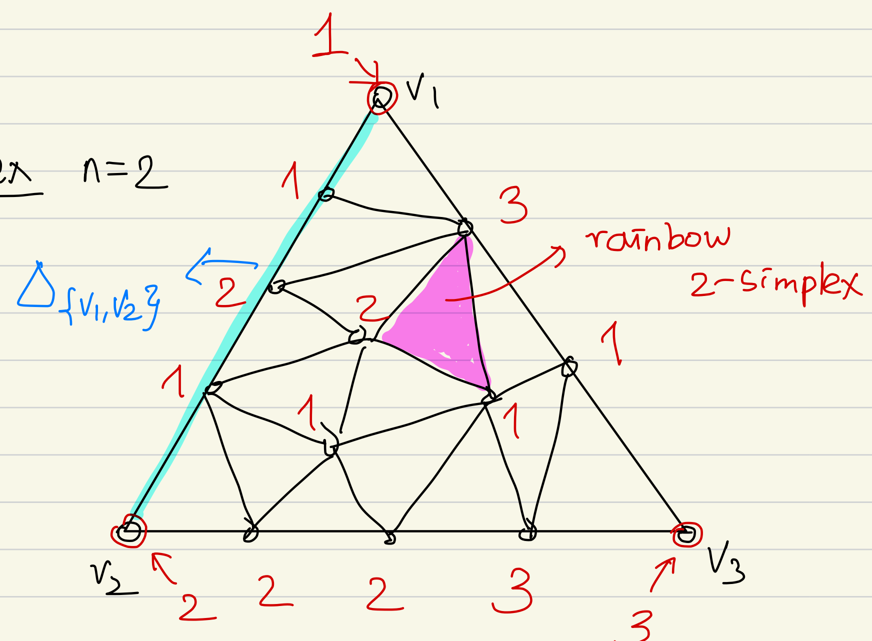
$e_1, \cdots, e_{n+1} \in \mathbb{R}^{n+1}$ standard basis $\quad e_i = (0, \cdots, 0, \overset{i\text{-th entry}}{1}, 0, \cdots, 0) \in \mathbb{R}^{n+1}$

$$\Delta^n = \{ a_1 e_1 + a_2 e_2 + \cdots + a_{n+1} e_{n+1} : a_i \geq 0, \ \Sigma a_i = 1 \}$$

ex $n=2$



ex $n=2$



- Δ^n : n-simplex on verticies $v_1, \cdots, v_{n+1}$
- T : a triangulation of Δ^n.

$f : V(T) \longrightarrow \{1, \cdots, n+1\}$ is a **Sperner coloring**

if ① $f(v_i) = i$ for $1 \leq i \leq n+1$

② If $v \in V(T) \cap \Delta_{\{v_{i_1}, \cdots, v_{i_k}\}}$, then $f(v) \in \{i_1, \cdots, i_k\}$

(weak)

Sperner's lemma : For any sperner coloring f of a triangulation T of Δ^n, there is an n-simplex of T which is rainbow

↳ colored by diff colors

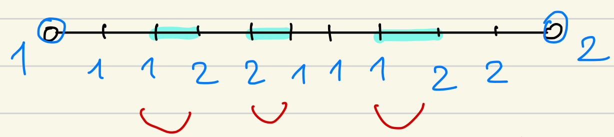
it has all colors $1, \cdots, n+1$.

* Stronger version : the # of rainbow n-simplex is odd

(pf of sperner's lemma)

By induction on n

$n=1$



↳ odd # of times

$\Rightarrow$ # of rainbow 2-simplicies is odd.

$n>1$ Using double counting!

$R :=$ # of rainbow n-simplicies of T (Goal : R is odd)

$Q :=$ # of n-simplicies of T having all of $1, \cdots, n$ as its color, but $n+1$.

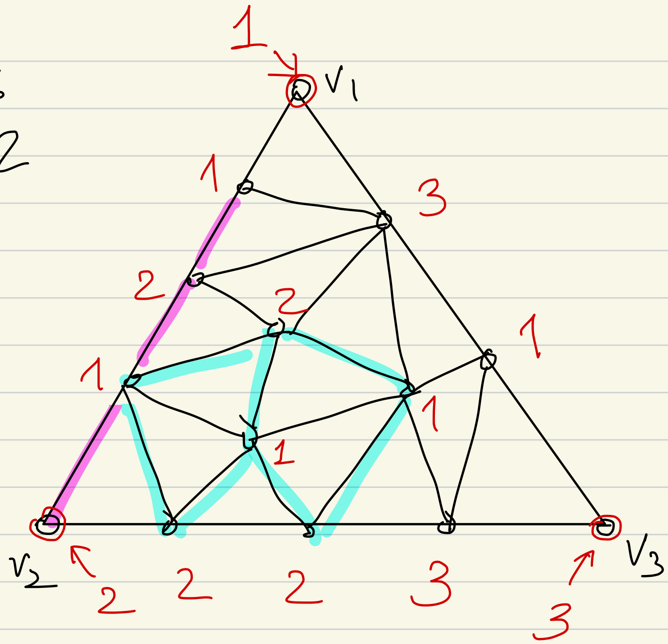
$1, 1, 2, 3, \cdots, n$ $\quad$ $1, 2, 2, 3, \cdots, n$

$X :=$ # of $(n-1)$-simplicies of T having all of $1, \cdots, n$ as its color that contained in $\boxed{\partial \Delta^n}$

$Y :=$ # of $(n-1)$-simplicies of T having all of $1, \cdots, n$ as its color that contained in $\mathrm{int}(\Delta^n)$

ex

$n=2$



$R=3$ $\quad X=3$

$Q=7$ $\quad Y=7$

1, 1, 2

1, 2, 2

$X+Y=10$

$R+2Q \neq$

- $R \rightarrow$ each rainbow n-simplex contains exactly one "$(n-1)$-simplex having all of $1,\cdots,n$ as its color"
- $Q \rightarrow$ each n-simplex of type Q contain exactly two "$(n-1)$-simplices having all of $1,\cdots,n$ as its color"

$R+2Q \neq$ # of $(n-1)$-simplicies having all of $1,\cdots,n$ as its color

$\parallel$

$X+2Y$

By the induction hypothesis, X should be an odd number. $\Rightarrow$ R is an odd number. ∎

$\left(\because f|_{V(T)\cap\Delta_{\{v_1,\cdots,v_n\}}}\right.$ is a Sperner coloring of $\left.\Delta^{n-1}\right)$

KKM theorem

Δ^n : n-simplex on verticies $1, \cdots, n+1$

A KKM covering is a collection of closed sets $A_1, \cdots, A_{n+1} \subseteq \Delta^n$

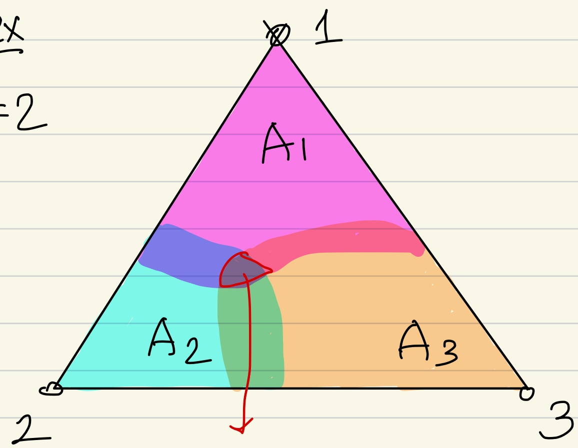
s.t. $\Delta_I \subseteq \bigcup_{i \in I} A_i$, $\forall I \subseteq [n+1]$

↳ if all of $A_1, \cdots, A_{n+1}$ are open sets, then same conclusion holds.

$$\boxed{\text{For every KKM covering } A_1, \cdots, A_{n+1} \text{ of } \Delta^n, \quad \bigcap_{i=1}^{n+1} A_i \neq \phi}$$

ex

$n=2$



$A_1 \cap A_2 \cap A_3 \neq \phi$

$A_1, A_2, A_3 \subseteq \Delta^2$

$I = \phi$

$I = \{1\}$

$\Delta_{\{1\}} \subseteq A_1$

$\Delta_{\{2\}} \subseteq A_2$

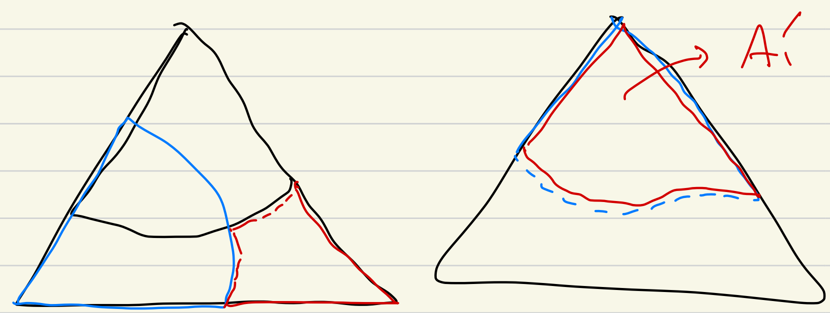
$\Delta_{\{3\}} \subseteq A_3$

$\Delta_{\{1,2\}} \subseteq A_1 \cup A_2$

$\Delta_{\{1,3\}} \subseteq A_1 \cup A_3$

$\Delta_{\{2,3\}} \subseteq A_2 \cup A_3$

$\Delta_{\{1,2,3\}} \subseteq A_1 \cup A_2 \cup A_3$



if some of A_i's are open and some of A_i's are closed, then it can be $\bigcap A_i = \phi$.

(pf of KKM thm) Using Sperner's lemma

pf sketch : Δ^n & KKM cover $A_1, \cdots, A_{n+1}$ $\rightsquigarrow$ a Sperner coloring

Take a sequence of triangulations of Δ^n : $T^1, T^2, \cdots, T^i, T^{i+1}, \cdots$

We will define a Sperner coloring.

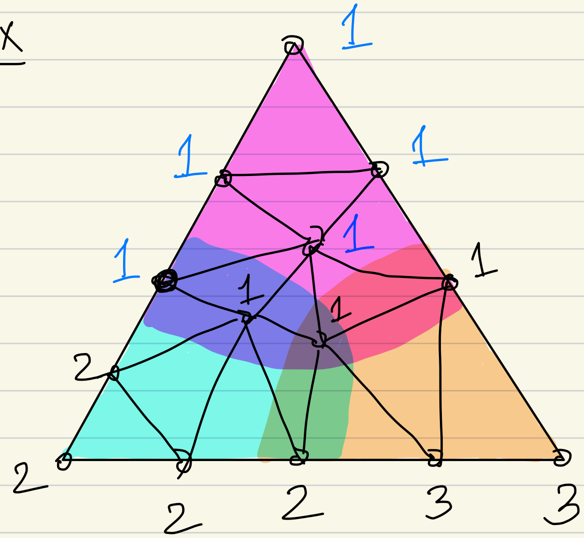
$T^i \underset{\text{subdivision}}{\hookrightarrow} T^{i+1}$ & $|T^i| \to 0$ as $i \to \infty$.

$|T^i|$: the max size of n-simplex of T^i

$f^i : V(T^i) \to \{1, \cdots, n+1\}$ is defined by $f^i(v) = \min j$ s.t. $v \in A_j$ & $j \in \operatorname{supp}(v)$.

$\operatorname{supp}(v) = \min\{I : v \in \Delta_I\}$

ex



$\operatorname{supp}(v) = \{1, 2\}$

$v \in \Delta_{\{1,2\}}$

Check ① f^i is well-defined

② f^i is a Sperner coloring

By Sperner's lemma, $\exists$ rainbow n-simplex σ^i of T^i

$\sigma^i = \{v_1^i, v_2^i, \cdots, v_{n+1}^i\}$

$\boxed{v_j^i \in A_j}$

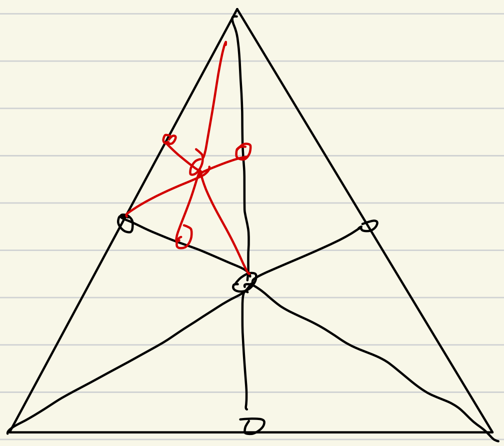
Sequence $\sigma^1, \sigma^2, \cdots$

↓ Bolzano — Weierstrass thm : every sequence in bdd & closed region has a convergent subsequence.

$\exists$ convergent subsequence $\sigma^{i_1}, \sigma^{i_2}, \cdots$

Since $\mathrm{vol}(\sigma^{i_j}) \to 0$ as $j \to \infty$, $\lim_{j\to\infty} \sigma^{i_j} = \{v\}$ for some v.

$\Rightarrow v \in \bigcap_{i=1}^{n+1} A_i$. ▨



$|T^i| \to 0$

$T^1 \to T^2 \to T^3$

KKM $\Rightarrow$ Brouwer fixed-point thm.

Brouwer fixed-point theorem

: $f: B^d \to B^d$ continuous map $\Rightarrow$ $x_0 \in B^d$ s.t $f(x_0) = x_0$.

$B^d := \{ x \in \mathbb{R}^d : \|x\| \le 1 \}$

(pf) Δ^n : n-ball. $\Delta^n = \{ (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \ge 0, \sum x_i = 1 \}$.

$f: \Delta^n \to \Delta^n$ continuous map. $\to$ Goal: $\exists x_0 \in \Delta^n$ s.t. $f(x_0) = x_0$.

* We will define a KKM cover using f.

For $x \in \Delta^n$, let $[x]_i$ be the i-th entry of x.

$A_i := \{ x \in \Delta^n : [x]_i \ge [f(x)]_i \}$ for each $i \in [n+1]$.

$\to$ closed set.

Check : $\{A_1, \cdots, A_{n+1}\}$ is a KKM cover of Δ^n.

For $I \subseteq [n+1]$, take $x \in \Delta_I$. Then $[x]_i = 0$, $i \notin I$.

$\Rightarrow \sum_{i \in I} [x]_i = 1$.

We want $x \in \bigcup_{i \in I} A_i$

Suppose not. Then $\boxed{[x]_i < [f(x)]_i}$ for all $i \in I$.

$\Rightarrow 1 = \sum_{i \in I} [x]_i < \sum_{i \in I} [f(x)]_i \leq 1.$ ※ contradiction.

By KKM theorem, $\exists\, x_0 \in \bigcap_{i=1}^{n+1} A_i$.

$\Rightarrow [x_0]_i \geq [f(x_0)]_i$ for all $i \in [n+1]$ $\Rightarrow x_0 = f(x_0)$. ∎

8/3 (Wed)

- $\boxed{\text{Brouwer} \Rightarrow \text{Sperner}}$

- Applications of KKM thm