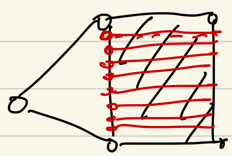
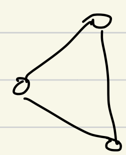
8/1 (Mon)

Reference : 〈Algebraic Topology〉 Hatcher

- Homotopy / Homotopy equivalence
- Simplicial complex / Homology group
- Nerve complex / Nerve theorem
- Topological Helly theorem

<u>Def</u> X, Y : topological spaces, $f_0, f_1 : X \to Y$.

- f_0, f_1 are <u>homotopic</u> if $\exists$ conti. map $F : X \times [0,1] \longrightarrow Y$ s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$, $\forall x \in X$. → We write $f_0 \simeq f_1$.
- A conti. map $f : X \to Y$ is called a <u>homotopy equivalence</u> if $\exists g : Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ → We write $X \simeq Y$.

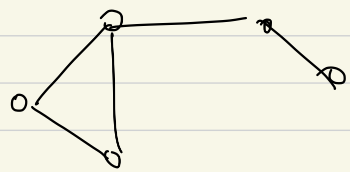
<u>ex</u> $X =$  $Y =$ 

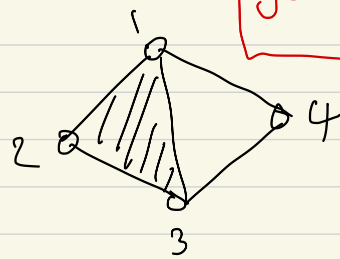
$X \simeq Y$

$f : X \to Y$ $\qquad$ $g : Y \to X$ inclusion map

exer $\boxed{\begin{array}{l} f \circ g = id_Y \\ g \circ f \simeq id_X \end{array}}$

Def (simplicial complex)





$\boxed{\text{geometric realization}}$

- X is an abstract simplicial complex on V if X is a collection of subsets of V that is closed under taking subsets.

  $(B \in X,\ A \subseteq B \Rightarrow A \in X)$

$$X = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}\}$$

- For $\sigma \in X$, $\dim(\sigma) = |\sigma| - 1$.

- $C_i(X;R) := \langle \sigma \in X : \dim \sigma = i \rangle_R$

  R: ring

  $C_i(X;\mathbb{F})$ ← vector space over $\mathbb{F}$.

  $\mathbb{F}$: field

$C_{-1}(X;\mathbb{F}) = \mathbb{F}$

$C_0(X;\mathbb{F}) \xrightarrow{\partial_0} C_{-1}(X;\mathbb{F})$ is defined by

$\partial_0([v]) = 1$

- $C_i(X;\mathbb{F}) \xrightarrow{\partial_i} C_{i-1}(X;\mathbb{F})$ is defined by

$$\partial_i([v_0, v_1, \cdots, v_i]) := \sum_{j=0}^{i} (-1)^j [v_0, v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_i]$$

extend linearly.

ex $\partial_2(\boxed{[1,2,3]}) = [2,3] - [1,3] + [1,2]$

$$[2,1,3] = -[1,2,3]$$
$$[2,3,1] = [1,2,3]$$

- $C_{i+1}(X;\mathbb{F}) \xrightarrow{\partial_{i+1}} C_i(X;\mathbb{F}) \xrightarrow{\partial_i} C_{i-1}(X;\mathbb{F})$

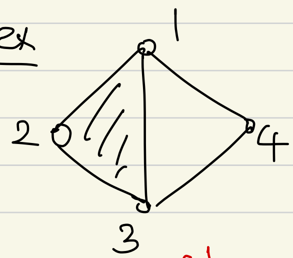
Check $\partial_i \circ \partial_{i+1} = 0$, i.e. $\operatorname{im}\partial_{i+1} \subseteq \ker \partial_i$

$\tilde{H}_i(X;\mathbb{F}) := \ker\partial_i / \operatorname{im}\partial_{i+1}$: The i-th homology group of X over $\mathbb{F}$.

$\hookrightarrow$ reduced homology group. (when we consider $C_{-1}(X;\mathbb{F})$)

Rmk $X \simeq Y \Rightarrow H_i(X;\mathbb{F}) \cong H_i(Y;\mathbb{F}),\ \forall i$

ex



$X = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}\}$

S^1

0-dim'l hole S^0 • • $\tilde{H}_0 \cong \mathbb{F}^1$

S^1 $\rightarrow$ $\tilde{H}_1 \cong \mathbb{F}^1$

$\tilde{H}_i = 0$

$$\tilde{H}_i(X;\mathbb{F}) = \begin{cases} \mathbb{F}^1 & i = 1 \\ 0 & i \neq 1 \end{cases}$$

$$0 \longrightarrow C_2(X;\mathbb{F}) \xrightarrow{\partial_2} C_1(X;\mathbb{F}) \xrightarrow{\partial_1} C_0(X;\mathbb{F}) \xrightarrow{\partial_0} C_{-1}(X;\mathbb{F}) \longrightarrow 0$$

$$\partial_2 = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & \\ [1,3] & \\ [1,4] & \\ [2,3] & \\ [3,4] & \end{array} \!\!\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

5×1

$$\partial_2([1,2,3]) = [2,3] - [1,3] + [1,2]$$

$$\partial_1 = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{c} \begin{array}{ccccc} [1,2] & [1,3] & [1,4] & [2,3] & [3,4] \end{array} \\ \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$\tilde{H}_i(X;\mathbb{F}) = 0 \quad \forall i \geq 3.$$

<u>exer</u>

$$\tilde{H}_0(X;\mathbb{F}) = 0.$$

$$\tilde{H}_2(X;\mathbb{F}) = \ker \partial_2 = 0$$

$$\tilde{H}_1(X;\mathbb{F}) = \ker \partial_1 / \operatorname{im} \partial_2 \cong \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \cong \mathbb{F}^1$$

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_4 = 0 \\ x_2 + x_4 - x_5 = 0 \\ x_3 + x_5 = 0 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 - x_3 \\ x_3 \\ x_1 \\ -x_3 \end{pmatrix} = x_1 \underbrace{\boxed{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}}_{\in \operatorname{im} \partial_2} + x_3 \boxed{\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{pmatrix}}$$

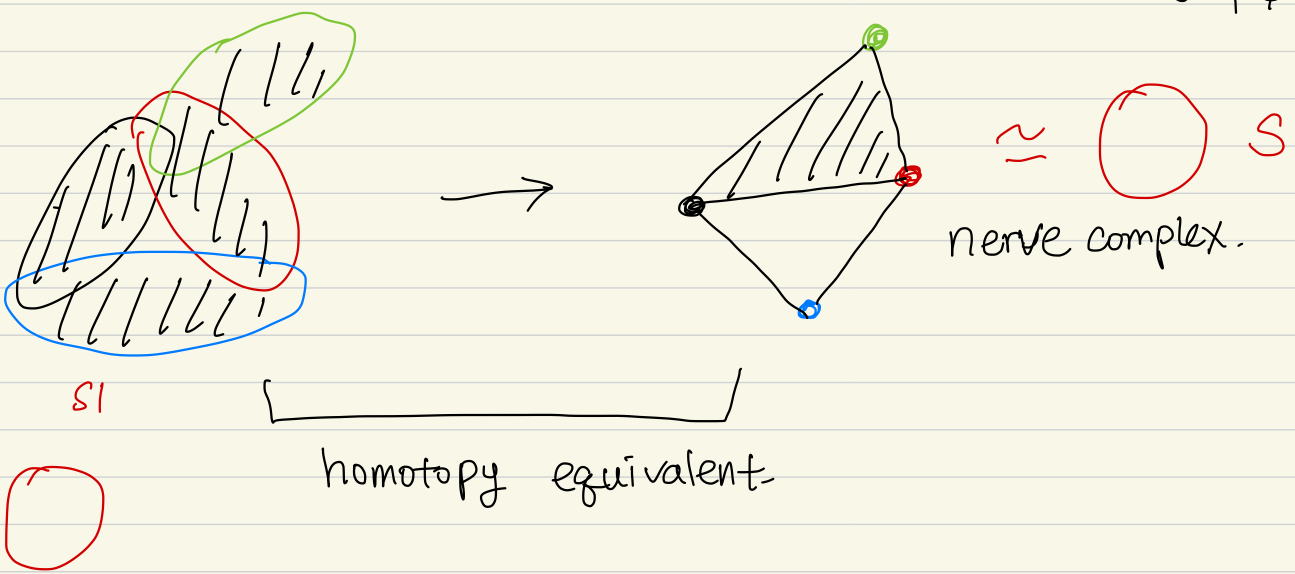
Nerve complex / Nerve theorem.

<u>Def</u> (Nerve complex)

ground set

$\mathcal{F}$: a family of sets $\longrightarrow$ $N(\mathcal{F}) := \{ \mathcal{F}' \subseteq \mathcal{F} ; \ \cap \mathcal{F}' \neq \phi \}$ nerve complex of $\mathcal{F}$.

$\cup \{ \phi \}$.



<u>Nerve theorem</u> For a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, $N(\mathcal{F}) \simeq \cup \mathcal{F}$

intersection of convex sets is a convex set.

$\hookrightarrow$ "contractible"

- A family $\mathcal{F}$ of sets in $\mathbb{R}^d$ is a <u>good cover</u> if any non-empty intersection of finitely many sets in $\mathcal{F}$ is contractible.

"Homological Nerve theorem"

<u>Nerve thm</u>: $\mathcal{F}$ is a good cover $\Rightarrow$ $\boxed{N(\mathcal{F}) \simeq \cup \mathcal{F}.}$

ex

$\leftarrow$ not a good cover

contractible.

Topological Helly theorem

Def (d-Leray)

X : a simplicial complex.

X is **d-Leray** if $\tilde{H}_i(Y;\mathbb{F}) = 0$ $\forall$ induced subcomplex Y of X, $\forall i \geq d$

- If $\mathcal{F}$ is good cover in $\mathbb{R}^d$, $N(\mathcal{F})$ is d-Leray.

($\because$ By Nerve thm, $N(\mathcal{F}) \simeq \bigcup\mathcal{F}$

$\bigcup\mathcal{F} \subseteq \mathbb{R}^d \Rightarrow \tilde{H}_i(\bigcup\mathcal{F};\mathbb{F}) = 0$ $\forall i \geq d$

$\Rightarrow \tilde{H}_i(N(\mathcal{F});\mathbb{F}) = 0$ $\forall i \geq d$.)

Topological Helly theorem

$\mathcal{F}$: a family of sets.

If $N(\mathcal{F})$ is d-Leray, then $\mathcal{F}$ has helly number at most $d+1$.

If any $(d+1)$ members of $\mathcal{F}$ have a point in common, then $\mathcal{F}$ has a point in common.

$\mathcal{F}$: a family of convex sets in $\mathbb{R}^d$

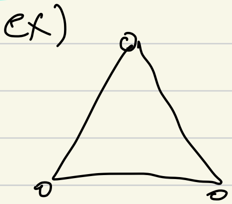
$\Rightarrow N(\mathcal{F})$ is d-Leray

(pf) Suppose not. $\exists\, F_1, \cdots, F_{k+1} \in \mathcal{F}$ s.t. any k of them are intersecting, but $\bigcap_{i=1}^{k+1} F_i = \phi$.

$k \geq d+1$

Consider $N(\{F_1, \cdots, F_{k+1}\}) \cong \partial \Delta_{\{F_1, \cdots, F_{k+1}\}} \simeq S^{k-1}$

ex)



2-simplex

$\partial(\Delta_2) \simeq S^1$

$\tilde{H}_{k-1}(N(\{F_1, \cdots, F_{k+1}\}) ; \mathbb{F}) \neq 0$

Since $k-1 \geq d$, this is a contradiction to d-Lerayness of $N(\mathcal{F})$. $\blacksquare$