8/4 Topological Combinatorics 2nd part - 1

<u>GOAL</u>
- Hall's theorem for hypergraphs
- Application - first case of Ryser's conjecture

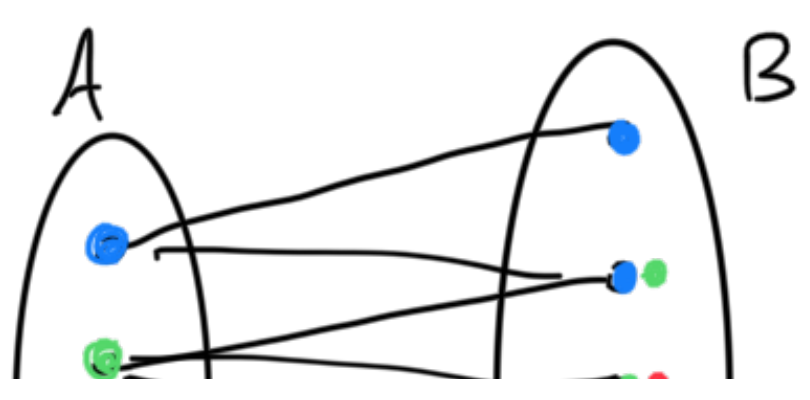
<u>Hall's theorem</u> X : ground set

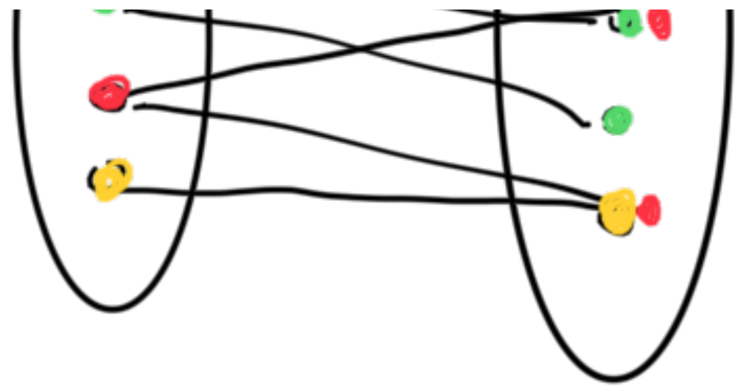
$X_1, X_2, \cdots, X_m \subseteq X$.

If $\forall I \subseteq [m] := \{1, 2, \cdots, m\}$ $(\neq \phi)$ $\left|\bigcup_{i \in I} X_i\right| \geq |I|$.

then there exists $\{a_1, a_2, \cdots, a_m\} \subseteq X$ s.t.

$a_1 \in X_1, a_2 \in X_2, \cdots, a_m \in X_m$ (rainbow set)





equivalent

Hall's THM (bipartite graph ver.) In a bipartite graph,

$\exists$ matching covering A if $\forall S \subseteq A$, $|N(S)| \geq |S|$

$\left(N(S) := \{ b \in B : b \text{ is adjacent to some } a \in S \} \right)$

Hypergraph $\mathcal{H}$ on V : $\mathcal{H} \subseteq 2^V$

$(A \in \mathcal{H}$ **edge**$)$

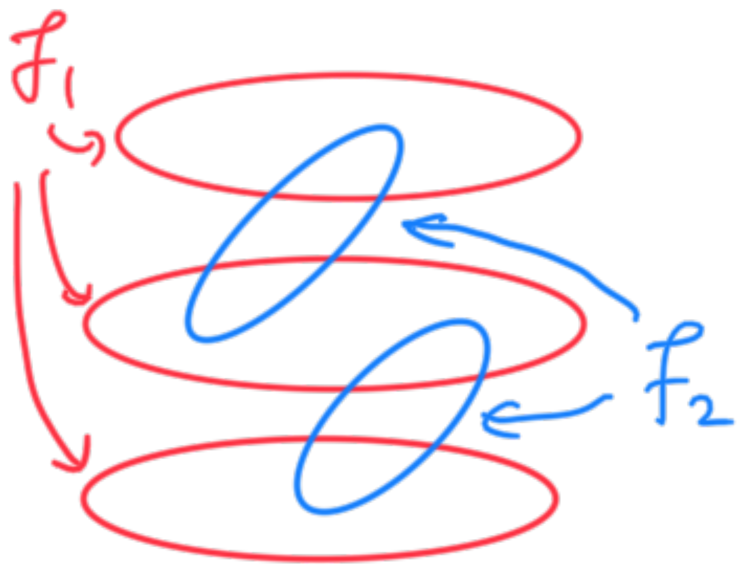
- $\mathcal{H}$ is **r-uniform** if $\forall A \in \mathcal{H}$, $|A| = r$.

(e.g. 2-uniform hypergraphs are graphs)

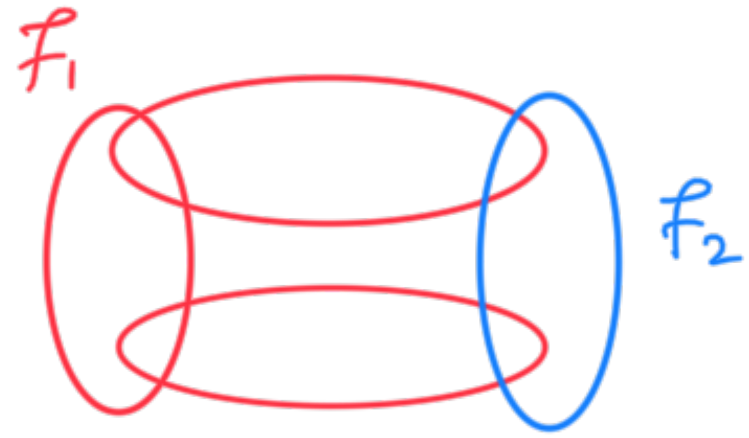
- A **matching** in $\mathcal{H}$ is a set of mutually disjoint edges
- $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{H}$. $\mathcal{F}_1$ **pins** $\mathcal{F}_2$ if

every edge of $\mathcal{F}$ meets some edge of $\mathcal{F}$

Every edge of $\mathcal{F}_2$ meets some edge of $\mathcal{F}_1$.



$\mathcal{F}_2$ pins $\mathcal{F}_1$



$\mathcal{F}_2$ does not pin $\mathcal{F}_1$

THM (Aharoni–Haxell, 2000)

$\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_m$ hypergraphs on V

If $\forall \, I \subseteq [m]$ $(\neq \emptyset)$, $\exists \, M_I$: matching in $\bigcup_{i \in I} \mathcal{H}_i$ that cannot be pinned by $< |I|$ edges of $\bigcup_{i \in I} \mathcal{H}_i$

then $\exists \, e_i \in \mathcal{H}_i$ s.t. $\{e_1, e_2, \cdots, e_m\}$ is a matching

(rainbow matching)

When each $\mathcal{H}_i$ is 1-uniform,
we can obtain Hall's theorem, because matching $\Leftrightarrow$ set of distinct vertices

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Proof is based on Sperner's LEMMA.

Outline
① Construct a "special" triangulation T of a simplex on m vertices
② label each vertex of T with edges from $\mathcal{H}_i$'s
③ Apply Sperner's lemma to obtain a "rainbow" simplex S
④ S corresponds to a rainbow matching.

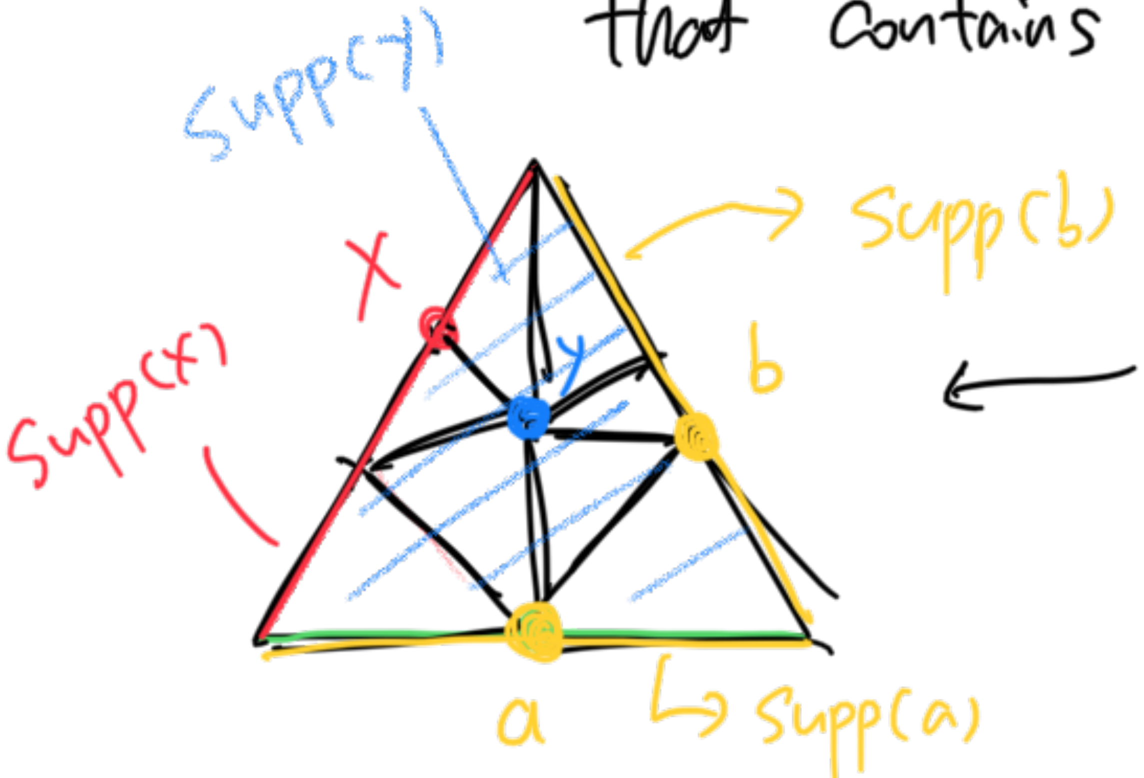
Special Triangulation.

Given a triangulation T of a simplex Δ

$\forall x$: point in T

Supp(x) : the unique face of Δ

that contains x in its relative interior

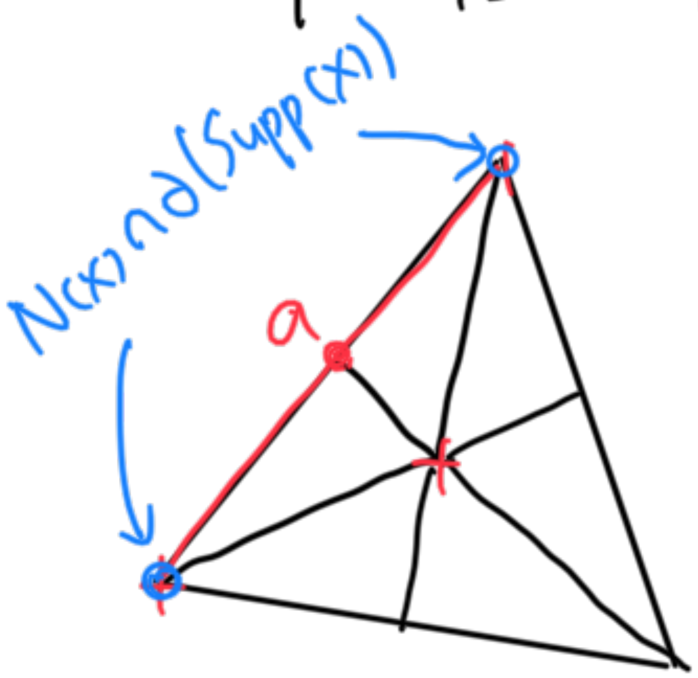


$\leftarrow$ Non-hierarchic triangulation.

- T is hierarchic if $\forall x, y \in V(T)$ that are adjacent,

either $\mathrm{Supp}(x) \subseteq \mathrm{Supp}(y)$

or $\mathrm{Supp}(y) \subseteq \mathrm{Supp}(x)$



$\leftarrow$ hierarchic

but not economically

- T is economically hierarchic

if in addition $\forall x$, $\partial(\mathrm{supp}(x)) \cap N(x)$ forms a simplex.

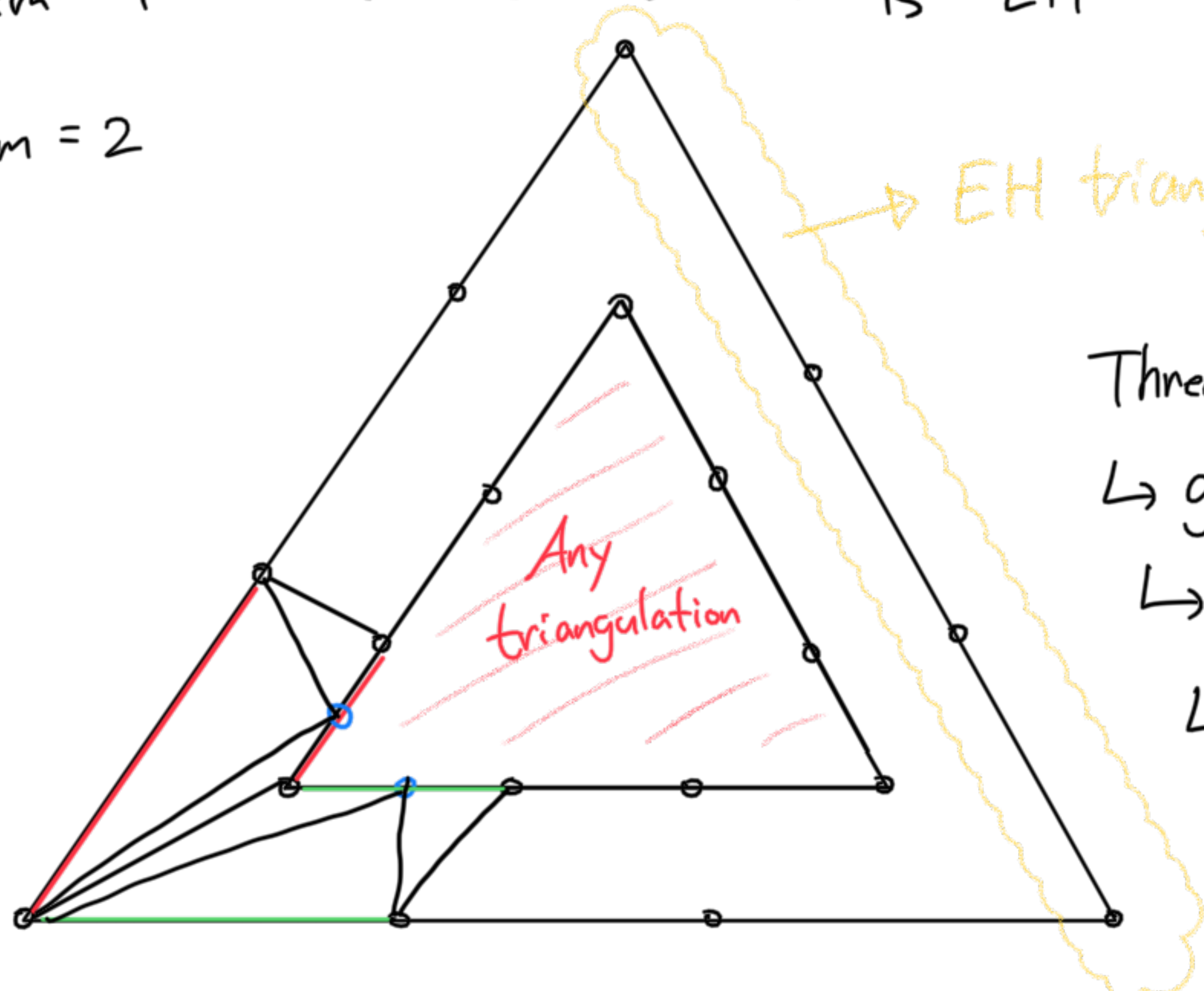
boundary

<u>LEMMA</u> Every simplex (in any dimension) has an economically hierarchic triangulation. (EH)

dim = 1

is EH

dim = 2



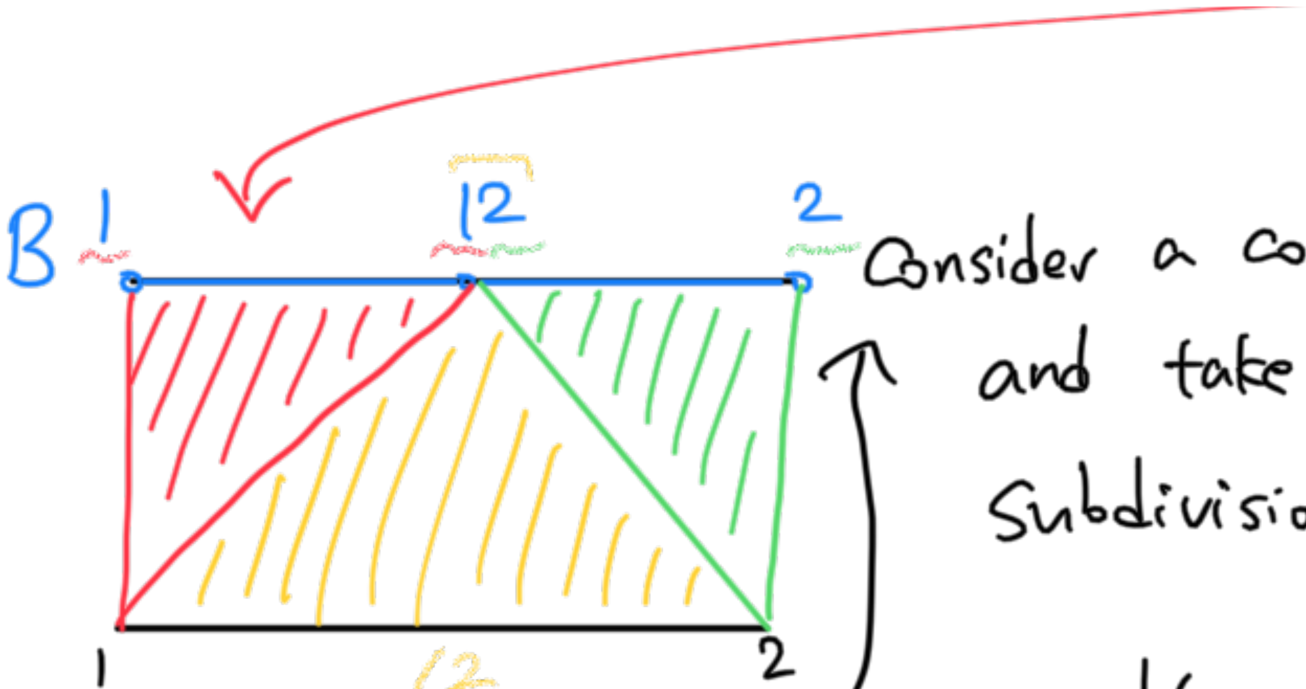
EH triangulation is 1-dim

Three copies of 1-dim EH triang.

↳ glue them to make a triangle A

↳ put a smaller homothet B in the interior of A

↳ $\forall$ simplex σ in A its corresponding simplex σ' in B put a triangulation



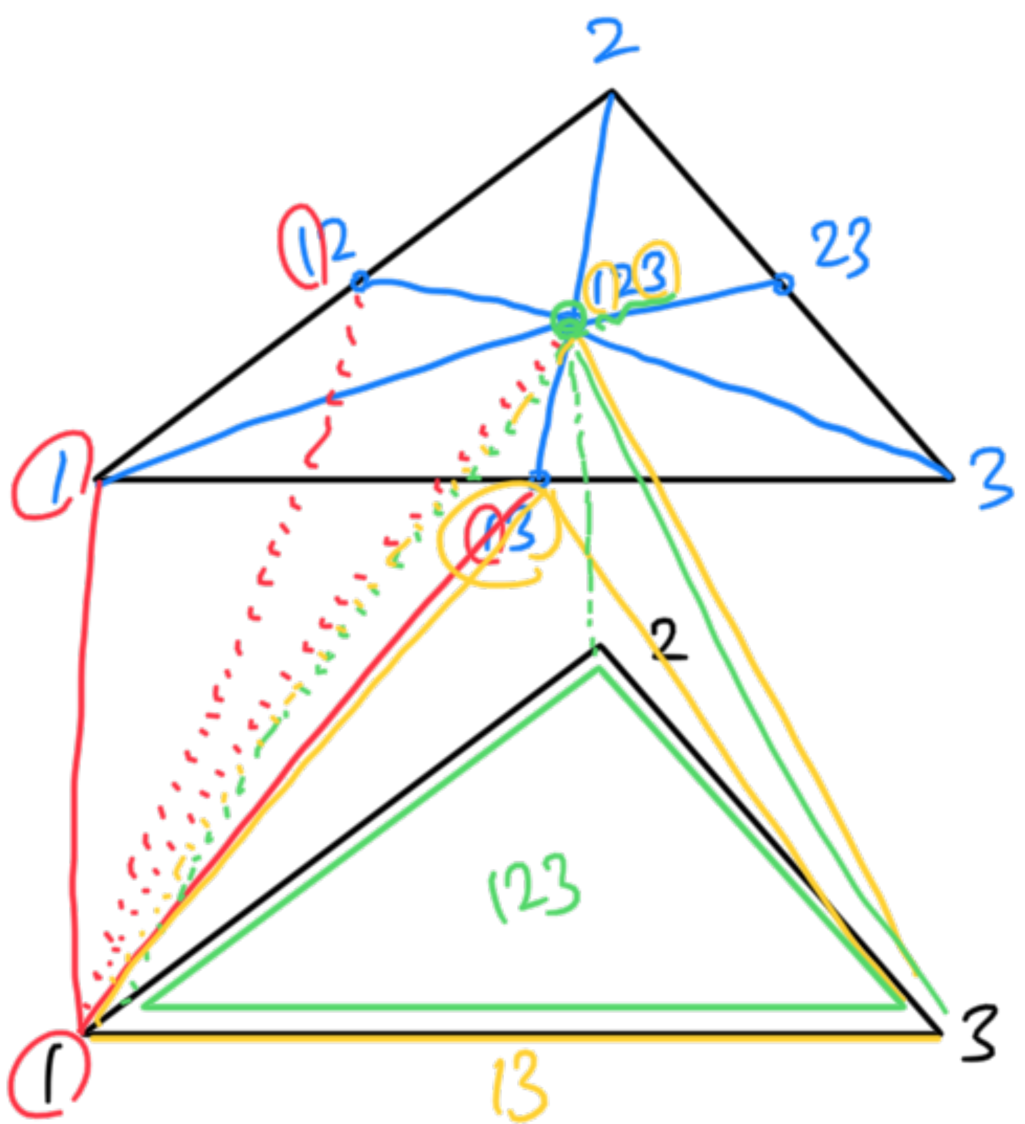
Consider a copy $S \times \{1\}$ and take a barycentric subdivision : say B.

For every simplex $S \times \{0\}$

$\forall \sigma \in S \times \{0\}$, $\tau \in B$ s.t. every vertex of τ contains σ,

put a simplex on $\sigma \cup \tau$

$\Rightarrow$ this gives a triangulation of $S \times [0,1]$



proof of Aharoni – Haxell

Δ : simplex on m vertices

T : EH triangulation of Δ

$\forall$ point x in T, label x with an edge $e(x)$ from $\underline{M_{\mathrm{Supp}(x)}}$

that is not pinned by the edges labelled on $N(x) \cap \partial(\mathrm{Supp}(x))$

$\Rightarrow$ $\forall$ adjacent pair x and y, either

(Check!) ① $e(x) = e(y)$ or ② $e(x)$ and $e(y)$ are disjoint.
(same color)

$\Rightarrow$ By Sperner, $\exists$ rainbow simplex.

↳ vertices are always Case ②.

$\Rightarrow$ rainbow matching.

Corollary $\mathcal{H}_1, \cdots, \mathcal{H}_m$ hypergraphs on V

$\forall I \subseteq [m]$ $(\neq \phi)$, $\exists M_I$ (matching) in $\bigcup_{i \in I} \mathcal{H}_i$ that is not pinned by $< |I| - d$ edges of $\bigcup_{i \in I} \mathcal{H}_i$

$\Rightarrow$ There is a rainbow matching possibly except for d colors.

$\left(\exists\, i_1, \cdots, i_{m-d} \text{ s.t. } \exists e_{i_j} \in \mathcal{H}_{i_j} \text{ where } \{e_{i_j} : 1 \le j \le m-d\} \text{ is a matching} \right)$

proof) Let $\mathcal{H}_i' = \mathcal{H}_i \cup \{ \underbrace{\{v_1\}, \{v_2\}, \cdots, \{v_d\}}_{\text{new vertices}} \}$ on $V \cup \{v_1, \cdots, v_d\}$,

Apply Aharoni–Haxell's thm to $\mathcal{H}_i'$s $\Rightarrow \exists$ rainbow matching

$\hookrightarrow$ possibly containing $\{v_i\}$'s.

Preview

König's thm : for every bipartite graph G

$$\nu(G) = \tau(G)$$

maximal size of a matching

minimal size of a vertex set that meets all edges.

Ryser's Conjecture for every r-partite r-uniform hypergraph $\mathcal{H}$

$$\tau(\mathcal{H}) \leq (r-1)\,\nu(\mathcal{H}).$$

(trivial : $\tau(\mathcal{H}) \leq r\,\nu(\mathcal{H})$)

$\hookrightarrow$ (r - some const) for $r=4,5$.

THM (Aharoni, 2001) Ryser's Conjecture is true for $r=3$