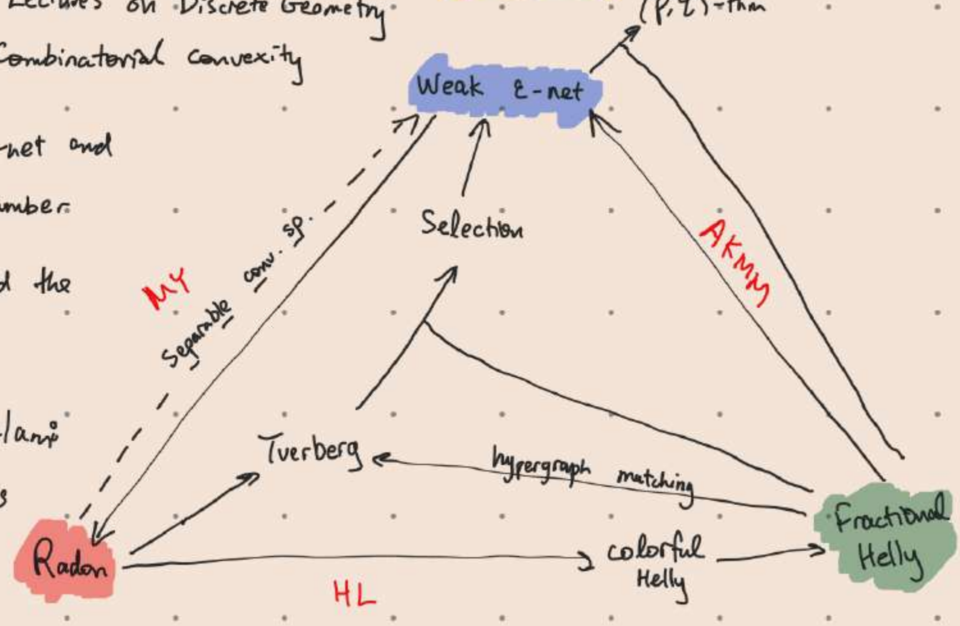
Lecture 6

$(X, \mathcal{C})$

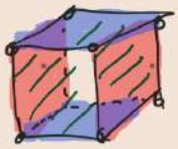
References

Books
- Matoušek: Lectures on Discrete Geometry
- Bárány: Combinatorial convexity

- Moran-Yehudayoff: On weak ε-net and the Radon number
- Holmsen-Lee: Radon numbers and the fractional Helly theorem
- Alon-Kalai-Matoušek-Meshulam: Transversal numbers for hypergraphs arising in geometry



1. Rmk: $h_{col}(\mathcal{C}) \leq S(r_h(\mathcal{C}), h(\mathcal{C}))$ Radon # $r_2(\mathcal{C})$ $n=3$



Tight when Helly # is 2: $S(r, 2) = 2^{r-1} - 1$

Consider subcubes $\{0,1\}^n$: $r \approx \log_2 n$, $h_{col} = n+1$

Q: How tight when $h \geq 3$?

2. Any other properties that can replace the role of fract. Helly?

3. Prove weak ε-net (or (p,q)-type) theorems for hypergraphs without bounded Radon number.

§ Separable convexity spaces

$(X, \mathcal{C})$, $\mathcal{C} \subseteq 2^X$

$A, B \in \mathcal{C}$, $\Rightarrow A \cap B \in \mathcal{C}$.

Given a conv. sp. $(X, \mathcal{C})$,

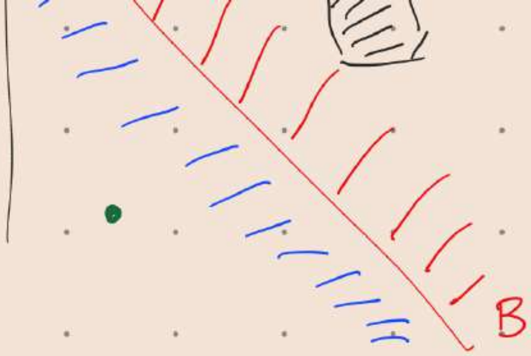
Def: a convex set $B \in \mathcal{C}$ is a half space if its complement is also in $\mathcal{C}$.

Write $\mathcal{B} \subseteq \mathcal{C}$ for the set of all half spaces

We say that $(X, \mathscr{C})$ is separable if

$\underbrace{\forall\, C \in \mathscr{C}}_{\text{convex set}}$ and $\forall$ pt $x \in X \setminus C$,

$\underbrace{\exists\, B \in \mathscr{B}}_{\text{half space}}$ separating them, i.e. $\begin{cases} C \subseteq B \\ x \notin B \end{cases}$

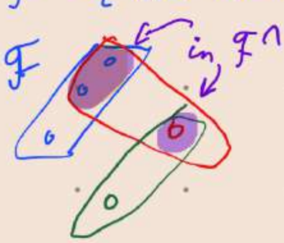


<u>Exer</u>: A convexity space $(X, \mathscr{C})$ is separable

$\iff$ every conv. set $C \in \mathscr{C}$ is the intersection of all half spaces containing it, i.e. $\mathscr{C} = \underline{\mathscr{B}^{\cap}}$

So $\mathscr{B}$ is a set of 'basis' that generate $\mathscr{C}$.

$\mathscr{F}^{\cap} = \{ A : A = \cap \mathscr{G},\ \mathscr{G} \subseteq \mathscr{F} \}$



<u>Exer1</u>. Let $(X, \mathscr{C})$ be a separable conv. sp w/ half spaces $\mathscr{B}$. Prove that the VC dim of $\mathscr{B}$ is less than Radon number of $\mathscr{C}$.

(seperability is needed here)

<u>Def</u>: Given a hypergraph $\mathscr{F}$ on vertex set X, a set of vertices $A \subseteq X$ is shattered by $\mathscr{F}$ if $\mathscr{F}|_A = 2^A$, i.e.

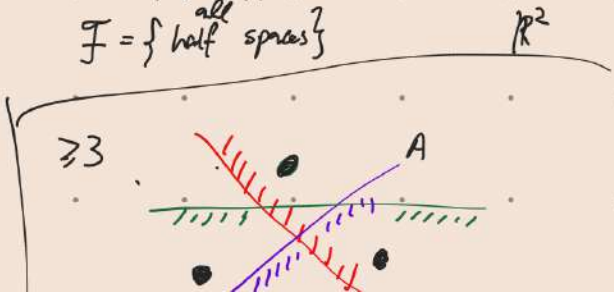
$\forall\ A' \subseteq A$, $\exists$ an edge $S \in \mathscr{F}$ s.t. $S \cap A = A'$

The VC dim of $\mathscr{F}$ is the cardinality of the largest set that can be shattered by $\mathscr{F}$.

<u>Ex</u>. Consider half spaces in $\mathbb{R}^2$, i.e. $X = \mathbb{R}^2$

$\mathscr{F} = \{\text{all half spaces}\}$

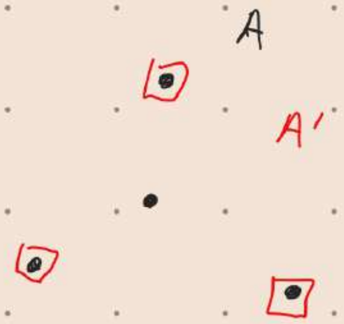
VC. dim of $\mathscr{F} = 3$



- ≥ 3

- < 4 i.e. need to find 4 pts that cannot be shattered by half spaces

<4



<u>Thm2</u> Let $\mathcal{F}$ ($\nearrow \mathcal{B}$) be a hypergraph on finite set X w/ $h(\mathcal{F}) = h$ and VC dim d

$$\Rightarrow \quad N(\varepsilon, \underbrace{\mathcal{F}^{\cap}}_{\mathcal{C}}) \le \left(\frac{120h^2}{\varepsilon}\right)^{4hd \log 1/\varepsilon}$$

<u>Rmk</u> Apply this to $\mathcal{B}$, $\mathcal{C} = \mathcal{B}^{\cap}$

- Levi: $r(\mathcal{C}) = r$, $h(\mathcal{C}) < r$
- <u>Exer1</u> $\Rightarrow$ VC dim of $\mathcal{B} < r(\mathcal{C})$

$\boxed{\begin{aligned}\mathcal{F}^{\cap} &= \mathcal{C} \\ \mathcal{F} &= \mathcal{B}\end{aligned}}$

§ Pf of Thm 2

Fix a fin. supp. meas. μ on X, we need to upp bd $N_\mu(\varepsilon, \mathcal{F}^{\cap})$.

Idea

(i) if $\varepsilon > 1 - 1/h$, then a Helly pt suffices as it pierces all $\mathcal{F}^{\cap}_{\varepsilon,\mu}$

(ii) for smaller ε, density increment via Haussler's packing lem.

<u>Illustration of one step of (ii)</u>

- Take a $C \in \mathcal{F}^{\cap}$ w/ $\mu(C) \ge \varepsilon > (1 - 1/h)^2$ we want to pierce.

  We may assume $C \subseteq B \in \mathcal{F}$ w/ $\mu(B) \le 1 - 1/h$, for otherwise (i) applies.

  $C = \cap \mathcal{F}'$ for some $\mathcal{F}' \subseteq \mathcal{F}$

- Now consider $\mu|_B$, the meas. conditioned on B $\left(\mu|_B(A) = \frac{\mu(A \cap B)}{\mu(B)} \right)$

We get a density increment: $\mu|_B(C) = \frac{\overset{\mu(C)}{\mu(C \cap B)}}{\mu(B)}$

B

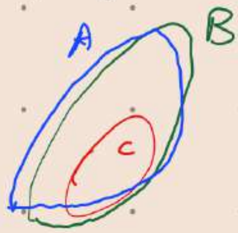
$$> \frac{(1-1/h)^2}{1-1/h} = 1-1/h$$

Then we can zoom into $\mu|_B$ and use (i) to find a single pt (Helly pt) piercing all such $C \subseteq B$. The problem is that we have no way to upp. bd # choices of B (could be as large as $|\mathcal{F}|$)

Here is where Haussler's packing lem kicks in, giving a collection $\mathcal{H}$ of bounded number of sets approximating those in $\mathcal{F}$. thanks bdd VC dim of $\mathcal{F}$.

<u>Thm</u> (Haussler's Packing lem) Let $\mathcal{F} \subseteq 2^X$ be a hyperg. w/ VC dim d. For any meas. μ on X and any $\delta > 0$, there exists $\mathcal{H} \subseteq \mathcal{F}$ s.t.

- $|\mathcal{H}| \leq \left(\frac{4e^2}{\delta}\right)^d$ and
- $\forall B \in \mathcal{F}$, $\exists A \in \mathcal{H}$ s.t. $\mu(A \triangle B) \leq \delta$



<u>Pf of thm 2</u>

- Let $\mathcal{L} \subseteq \mathcal{F}$ be the subfam. of all large $B \in \mathcal{F}$ w/ $\mu(B) > 1 - 1/h$.

  Helly # $= h \Rightarrow \exists$ a pt x piercing $\mathcal{L}$, hence $\mathcal{L}^\cap$.

- Take $0 < \varepsilon < 1$. We shall construct a weak ε-net $S = S(\varepsilon, \mu)$ by induction on $m(\varepsilon) = \min \left\{ t \in \mathbb{N} \cup \{0\} : \varepsilon \left(1 + \frac{1}{2h}\right)^t > 1 - 1/h \right\}$

  <u>Base case</u>: $m(\varepsilon) = 0$, then $\varepsilon > 1 - 1/h$ and $S = \{x\}$ suffices as every $C \in \mathcal{F}^\cap$ w/ $\mu(C) \geq \varepsilon > 1 - 1/h$ is in $\mathcal{L}^\cap$.

Inductive step: $m(\varepsilon) > 0$ Set $\delta = \frac{\varepsilon}{(2h)^2}$, $\varepsilon' = \left(1 + \frac{1}{2h}\right)\varepsilon$

Note that $m(\varepsilon') = m(\varepsilon) - 1$

Take $\mathcal{H} \subseteq \mathcal{F}$ from Haussler's packing lem, $|\mathcal{H}| \leq \left(\frac{4e^2}{\delta}\right)^d$

For each $A \in \mathcal{H}$ w/ $\mu(A) > 0$, by induction hypothesis,

we get a piercing set $S_A = S(\varepsilon', \mu|_A)$.

We claim that

$$S = \{x\} \cup \bigcup_{A \in \mathcal{H} : \mu(A) > 0} S_A \quad \text{is what we need.}$$

- To see that S pierces all $C \in \mathcal{F}^\cap$ w/ $\mu(C) \geq \varepsilon$, we may assume $C \notin \mathcal{L}^\cap$, o.w. $x \in C$.

Thus, $C \subseteq B \notin \mathcal{L}$, i.e. $\mu(B) \leq 1 - 1/h$.

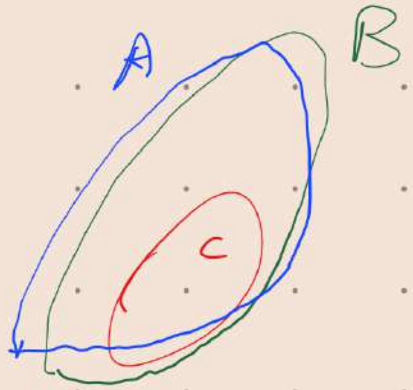
- Let $A \in \mathcal{H}$ be the one approx. B: $\mu(A \triangle B) \leq \delta$.

It suffices to show $\underline{\mu|_A(C) \geq \varepsilon'}$ as then C is pierced by $S_A = S(\varepsilon', \mu|_A)$

Indeed,

$$\mu|_A(C) = \frac{\overbrace{\mu(C \cap A)}^{\geq \mu(C) - \mu(A \triangle B)}}{\underbrace{\mu(A)}_{\substack{\leq \mu(B) + \delta \\ \leq 1 - 1/h + \delta}}} \geq \frac{\mu(C) - \delta}{1 - 1/h + \delta}$$

$$\geq \frac{\varepsilon - \delta}{1 - 1/h + \delta} \geq \varepsilon'.$$



We are left to bdd $|S|$ by some funct. $f(m)$, where $m = m(\varepsilon)$

The above argument gives $f(0) = 1$

$$\underline{f(m)} \le \underline{1} + |\mathcal{H}|\, f(m-1) \le 1 + \left(\frac{4e^2}{\delta}\right)^d f(m-1) \quad \cdots \quad \blacksquare$$

$$\underbrace{S}_{\varepsilon\text{-net}} = \underline{\{x\}} \cup \bigcup_{\substack{A \in \mathcal{H} \\ \mu(A)>0}} \underline{S_A} \rightsquigarrow \varepsilon'\text{-net}$$

$$m(\varepsilon') = m(\varepsilon) - 1$$