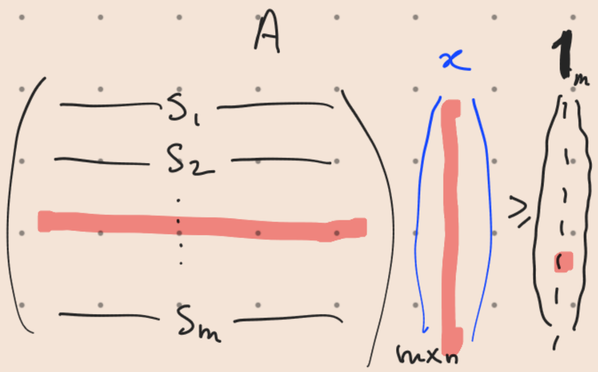
Lecture 19

§ Fractional transversals and fractional matchings.

Consider a hypergraph / set system $F = \{S_1, \dots, S_m\}$, $\forall i \; S_i \subseteq [n]$

Start by writing transversals algebraically.

Let A be the $m \times n$ 0/1-matrix



$\Rightarrow$ a transversal T can be viewed as a 0/1-vector $x \in \{0,1\}^n$ (i.e. $x = \mathbb{1}_T$)

s.t. $A \cdot x \geq \mathbb{1}_m$ (ineq holds coordinatewise)

b/c $(Ax)_i = |S_i \cap T| \geq 1$

$$\min \; x \cdot \mathbb{1}_n$$
$$\text{s.t. } Ax \geq \mathbb{1}_m$$
$$x_i \in \{0,1\} \longrightarrow x_i \in [0,1]$$

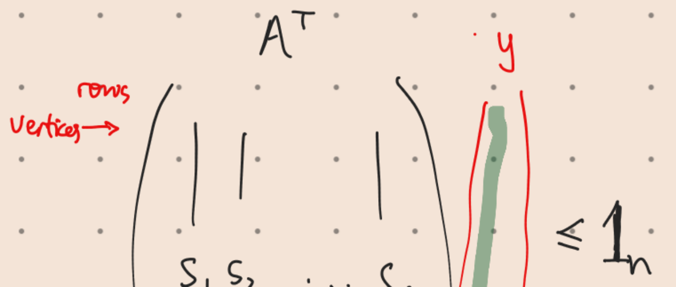
The optimal value of this linear program (LP) is the fractional transversal number of F, denoted by $\tau^*(F)$.

Rmk: It becomes computational much easier to calculate when relaxed to fractional version.

Obs: $\tau^*(F) \leq \tau(F)$.

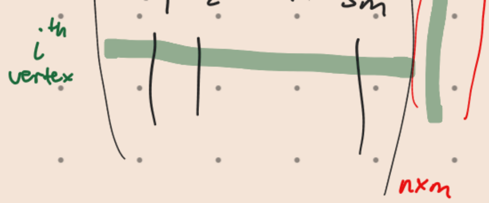
- Similarly, we can view a matching M in F as follows.

$$\max \; y \cdot \mathbb{1}_m$$
$$\text{s.t. } A^T y \leq \mathbb{1}_n$$
$$y_i \in \{0,1\} \longrightarrow y_i \in [0,1]$$



b/c $(A^T y)_i = \# S_j$ in y s.t. $i \in S_j$

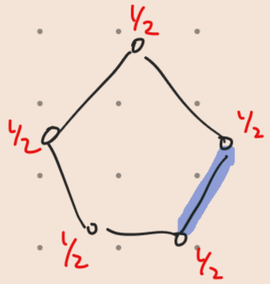
$y = \mathbb{1}_M$



The maximum of this LP is the fractional matching # of F denoted by $\nu^*(F)$.

Obs $\nu(F) \le \nu^*(F)$

Ex 1



$$2 = \nu(C_5) \le \nu^*(C_5) = 5/2 = \tau^*(C_5) \le \tau(C_5) = 3$$

Ex 2 $F = \begin{pmatrix} [n] \\ n/2 \end{pmatrix}$

$$2 = \nu(F) \le \nu^*(F) = 2 = \tau^*(F) \le \tau(F) = n/2 + 1$$

For each $i \in [n]$ assign weight $2/n$

gap large

- For finite hypergraphs F: $\boxed{\nu(F) \le \nu^*(F) = \tau^*(F) \le \tau(F)}$

gap could be also arb. large

§ LP duality

consider F = n lines in general position

$\nu(F) = 1$, $\nu^*(F) = n/2$

each edge gets weight 1/2

Thm $\forall$ finite hypergraph F, $\nu^*(F) = \tau^*(F)$.

Moreover, this optimal value is a rational number and there exists an optimal fractional transversal and an optimal fract. matching w/ rational entries.

Thm (Duality of L Programming) Let A be an $m \times n$ real matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Let $P = \{ x \in \mathbb{R}^n : x \ge 0, \ Ax \ge b \}$ (transversal)

and $D = \{ y \in \mathbb{R}^m : y \ge 0, \ y^T A \le c^T \}$ (matching)

If both $P \ne \emptyset$ & $D \ne \emptyset$, then

$$\min\{ c^T x : x \in P \} = \max\{ y^T b : y \in D \}.$$

§ Pf of (p,q)-thm via LP duality.

$$\underbrace{\nu(F) \overset{\text{fractional Helly}}{\le} \nu^*(F)}_{(\mathrm{II})} \overset{\text{LP duality}}{=} \underbrace{\tau^*(F) \overset{\text{weak } \varepsilon\text{-net}}{\le} \tau(F)}_{(\mathrm{I})}$$

Recall in $(p, d+1)$-thm, F: a fin. fam. conv. sets in $\mathbb{R}^d$ w/ $(p, d+1)$ property.

Goal: $\tau(F) \le g(p,d)$

Rmk: Note that $\nu(F) < p$ because of $(p, d+1)$-property.

So in (p,q)-thm, we try to bound the integrality gap here

Rmk: We can think of F as a finite hypergraph as follows.

- $\mathcal{H} = \{G \subseteq F : \cap G \neq \phi\}$, $\forall\, G \in \mathcal{H}$, pick $p_G \in \cap G$.

$V(F) = \{p_G : G \in \mathcal{H}\}$, $\forall\, C \in F \xrightarrow[\text{to the edge}]{\text{corresponds}} \{p_G : p_G \in C\}$

Part(I) is another formulation of weak ε-net thm.

Thm: $\forall$ finite F fam. conv. sets in $\mathbb{R}^d$

$\Rightarrow$ $\tau(F) \le f(1/\tau^*(F), d)$, where $f(\varepsilon, d)$ is the size of weak ε-net for pts in $\mathbb{R}^d$.

Pf: Idea: use fractional transversal to construct a set X s.t. F is $\frac{1}{\tau^*(F)}$-chubby w.r.t. X, then apply weak ε-net thm.

- Consider an optimal fract. transversal:

$$p \in V(F) \longleftrightarrow \text{weight } \frac{m_p}{D}, \quad m_p, D \in \mathbb{N}$$

so
$$\begin{cases} \displaystyle\sum_{p \in V(F)} \frac{m_p}{D} = \tau^*(F) & \cdots\cdots (1) \\ \displaystyle\forall\, C \in F: \sum_{p \in C} \frac{m_p}{D} \ge 1 & - - - - (2) \end{cases}$$

- Let X be a blowup of $V(F)$, where the multiplicity of each p is m_p

Left to show F is $\frac{1}{\tau^*(F)}$-chubby w.r.t. X

i.e. $\forall C \in F$, $|C \cap X| \geq \frac{1}{\tau^*(F)} \cdot |X|$.

- $|X| = \sum_{p \in V(F)} m_p \overset{(1)}{=} \tau^*(F) \cdot D$

- $\forall C \in F$, $|C \cap X| = \sum_{p \in C} m_p \overset{(2)}{\geq} D = \frac{1}{\tau^*(F)} \cdot |X|$ ∎

This thm $\Rightarrow$ $\tau(F) \leq$ a funct. of $(\underset{\underset{\nu^*(F)}{\parallel}}{\tau^*(F)}, d)$

So to prove (p,q)-thm, left to bound $\nu^*(F)$ by a funct. of (p, d).

Part (II)

Thm F fin. fam. conv. sets in $\mathbb{R}^d$

$(p, d+1)$-property $\Longrightarrow$ $\nu^*(F) \leq g(p, d)$

Pf: • Take an optimal fractional matching of F:

$$C \longmapsto \text{weight } \frac{m_C}{D}, \quad m_C, D \in \mathbb{N}$$

So
$$\begin{cases} \sum_{C \in F} \frac{m_C}{D} = \nu^*(F) & \text{------ (1)} \\ \forall p \in V(F) : \sum_{C : p \in C} \frac{m_C}{D} \leq 1 & \text{---- (2)} \end{cases}$$

• Shall apply fract. Helly to blowup of F.

Recall $\forall$ blowup F^* of F has $(p', d+1)$-property, where $p' = (p-1)d+1$.

So fract Helly $\Rightarrow$ F^* has an intersecting subfam. of size $\beta |F^*|$

where $\beta = \beta(\alpha, d) = \beta(p, d)$, $\alpha = \frac{1}{\binom{p'}{d+1}}$

• Let F^* be the blowup of F where multiplicity of C is m_C.

$$|F^*| = \sum_{C \in F} m_C \overset{(1)}{=} \nu^*(F) \cdot D$$

fract. Helly $\Rightarrow$ $\exists\, p \in V(F^*) = V(F)$ s.t.

edges of F^* containing p $= \sum_{C:\, p \in C} m_C \geq \beta\, |F^*| = \beta \cdot \nu^*(F) \cdot D$

• On the other hand by (2)

$$\sum_{C:\, p\in C} m_C \leq D$$

$$\Rightarrow \quad \nu^*(F) \leq 1/\beta = g(p,d) \qquad \blacksquare$$