Lecture 18

Recall :

$p \geq q > d$ $\qquad\qquad$ $HD_d(p,q)$

- (p,q)-thm : F fin. conv sets in $\mathbb{R}^d$ $\quad (p,q)$-property $\Rightarrow \tau(F) \leq f(p,q,d)$ $\;(= HD_d(p,q))$

Suffices to $(p, d+1)$-thm

Ingredients $\quad \mathbb{R}^d$

(i) Main 1. Fractional Helly : positive fract. of $(d+1)$-tuples in F intersecting $\Rightarrow$ positive fract. of F intersecting

(ii) Main 2. Weak ε-net : F is ε-chubby w.r.t. X $\Rightarrow \tau(F) \leq f(\varepsilon, d)$

$(\forall C \in F,\ |C \cap X| \geq \varepsilon |X|)$

(iii)

☆ Key property of $(p, d+1)$-property : "hereditary" in the following sense.

F fin fam. $\mathbb{R}^d$ w./ $(p, d+1)$-property $\Longrightarrow$ any of its blowup F^* has $(p', d+1)$-property

indep. of n.

where $p' = (p-1)d + 1$.

Consequence : $\forall$ blowup F^*, because of $(p', d+1)$-property

positive fract. of $(d+1)$-tuples in F^* intersecting (so fract. Helly applies to F^*)

$\Rightarrow$ # intersecting $(d+1)$-tuples in F^* $\geq$ $\binom{n}{p'} \Big/ \binom{n-d-1}{p'-d-1} = \frac{1}{\binom{p'}{d+1}} \binom{n}{d+1}$ $\qquad$ let $|F^*| = n$

<u>Pf</u> ($(p, d+1)$-thm)

Idea is to find a set of pts X s.t. F is ε-chubby w.r.t. X for some $\varepsilon = \varepsilon(p,d)$ $(\forall C \in F, |C \cap X| \geq \varepsilon |X|)$

then $\xRightarrow{\text{weak } \varepsilon\text{-net}}$ $\tau(F) \leq f(\varepsilon, d)$ bounded

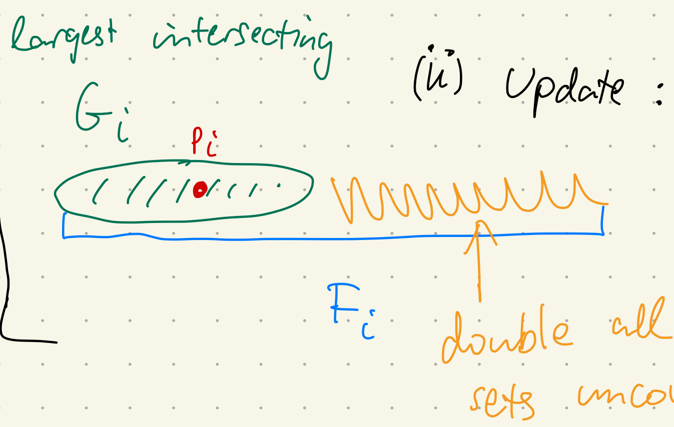
- Construct iteratively $X_0, X_1, \dots, X_t = X$

<u>Alg</u>

- Start w/ $F_0 = F$, $X_0 = \phi$
- For $i = 0, 1, \dots, t-1$: (i) Choose a largest intersecting subfam. $G_i \subseteq F_i$ pierced by some pt $p_i \in \mathbb{R}^d$.

  (ii) Update: $F_{i+1} = G_i \cup 2(F_i \setminus G_i)$

  $X_{i+1} = X_i \cup \{p_i\}$



It suffices to show that F is ε-chubby w.r.t. $X = X_t$.

<u>Observe</u>: • $|X| = |X_t| = t$

• Let $n := |F|$, $|F_{i+1}| \leq (2-\beta)|F_i|$

$\beta(\alpha, d) = \beta(p, d)$

$\alpha = \frac{1}{\binom{p'}{d+1}}$

where $p' = (p-1)d+1$

<u>pf</u>: • F_i is a blowup of F $\Rightarrow$ F_i has $(p', d+1)$-property, so Fract. Helly applies $\Rightarrow \exists\, \beta |F_i|$ size intersecting subfamily.

$$\Rightarrow \quad |F_t| \le (2-\beta)^t |F_0| = (2-\beta)^t \cdot n$$

- Take an arbitrary $C \in F$, let $k := |C \cap X|$

$\underline{\text{NTS}}$: $\quad k \geq \varepsilon \cdot |X| = \varepsilon \cdot t \quad$ for some $\varepsilon = \varepsilon(p, d)$

($\heartsuit$): if C does not contain p_i, then multiplicity of C gets doubled in F_{i+1}.

$\Longrightarrow$ so # copies of C in F_t

$$= 2^{t-k} \le |F_t| \le (2-\beta)^t \cdot n$$

$$\Rightarrow \quad t - k \le t \log_2(2-\beta) + \log_2 n$$

$$\Rightarrow \quad k \geq [1 - \log_2(2-\beta)]\, t - \log_2 n$$

want $\geq \varepsilon \cdot t$

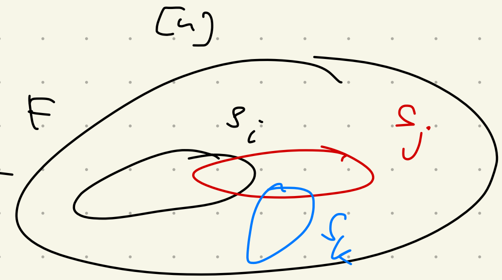
Choose t so that $\log_2 n \le \frac{1}{2}\left(1 - \log_2(2-\beta)\right) t$

$\left(\text{we can take e.g. } t = \dfrac{2}{1-\log_2(2-\beta)} \cdot \log_2 n\right)$

So

$$\geq \underbrace{\frac{1}{2}\left(1-\log_2(2-\beta)\right)}_{\varepsilon} t$$

Rmk: This doubling argument works largely because of the fact that $(p, d+1)$-property is "hereditary" in the sense that it's (weakly)
$(p', d+1)$-property
retained by blowups.



set system / hypergraph

$$\mathcal{F} = \{ S_1, \dots, S_m \}, \quad S_i \subseteq [n]$$

Def: $\tau(\mathcal{F})$: transversal number

min size transversal T

$\forall S_i \in \mathcal{F}, \ T \cap S_i \neq \emptyset$

$\nu(\mathcal{F})$: matching number

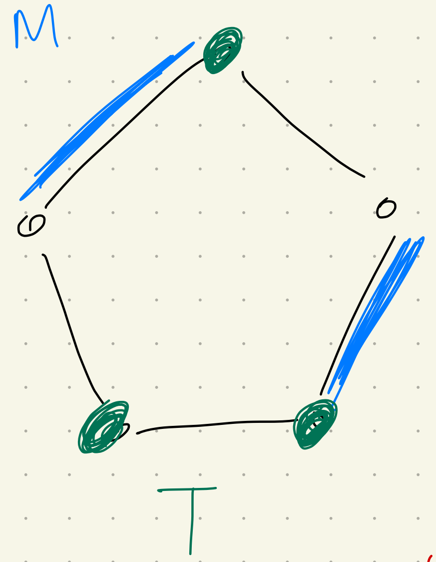
max size matching $M \subseteq \mathcal{F}$

$\forall S_i \neq S_j \in M, \ S_i \cap S_j = \emptyset$

Obs: $\nu(\mathcal{F}) \leq \tau(\mathcal{F})$

b/c for every edge/set in a max size matching (which has size $\nu(\mathcal{F})$) we need a distinct vertex to pierce.

- **Ex 1**



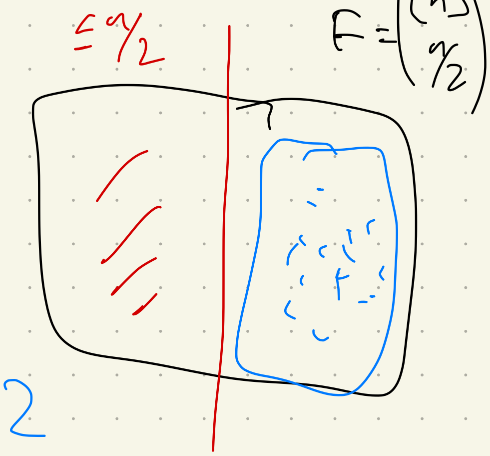
$\nu(C_5) = 2$

$\wedge$

$\tau(C_5) = 3$

gap could be arbitrarily large

- **Ex 2** $\quad F = \binom{[n]}{n/2}$

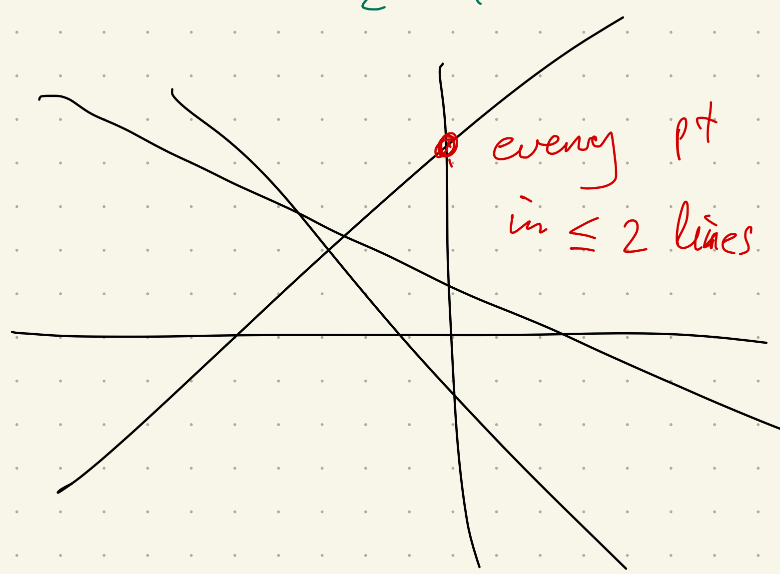


$\nu(F) = 2$

$\tau(F) \geq \frac{n}{2} + 1$

Geometric example

- Ex 3



F: n line in general positions

$\nu(F) = 1$

$\tau(F) \geq n/2$

Next: fractional version

LP-duality

$$\nu(F) \leq \nu^*(F) = \tau^*(F) \leq \tau(F)$$