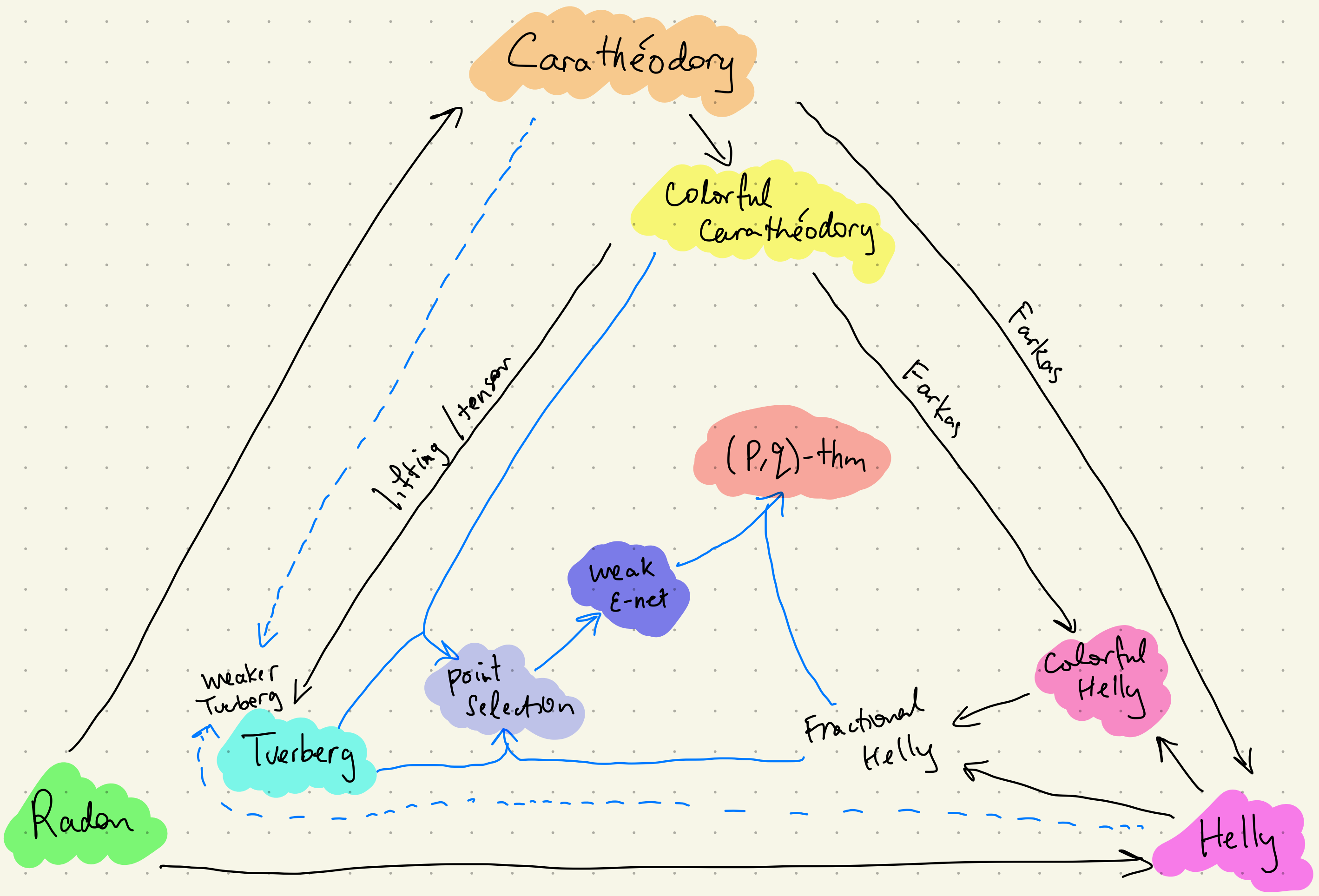
Lecture 17



§ (p,q)-thm.

Recall:

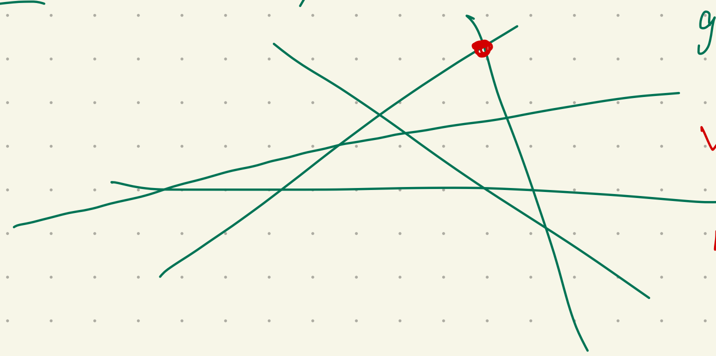
Defn A finite fam. of sets has (p,q)-property if every p-tuple contains an intersecting q-tuple.

Thm. $\forall\ p \geq q > d \geq 1$, $\exists\ HD_d(p,q)$ s.t. T.F.H.

If F is a finite fam. of convex sets in $\mathbb{R}^d$ w/ (p,q)-property, then it can be pierced by $\leq HD_d(p,q)$ pts.

Rmk: Need $q > d$: If only (d,d)-property, i.e. every d-tuple is intersecting, then no long gurantee to be pierced by bdd # pts.

Ex $d=2$, consider F = n lines in general position



✓ • (2,2)-property

need $\geq n/2$ pts to pierce F as no pt lies in 3 lines in F.

- Example of fam. w./ (p,q)-property

  ⊛ Start w./ $(4,3)$-property

  X : n pts in general position in $\mathbb{R}^2$

  Consider $F_\alpha = \{ \operatorname{conv} Y : Y \subseteq X, |Y| \geq \alpha \cdot |X| \}$

  w./ $\alpha > 1/2$.

  $\forall$ 4-tuple (C_1, C_2, C_3, C_4): $\sum_{i \in [4]} |C_i \cap X| \geq 4\alpha |X|$

  $> 2|X|$

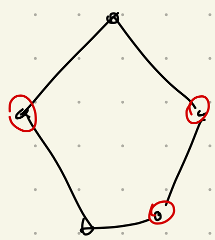
  $\Rightarrow$ Some pt in X appears in ≥ 3 sets in $C_1, \dots, C_4$

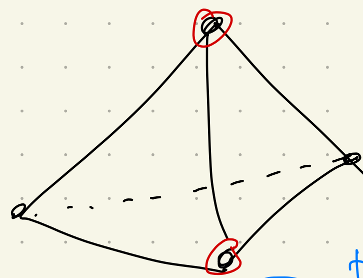
  ⊛ In general, take $\alpha > \frac{q-1}{p}$, any finite fam F that is α-chubby w.r.t. some set X

  $\Rightarrow$ F has (p,q)-property

- Connection to χ-boundedness in graph theory

<u>Defn</u> The transversal number of a hypergraph H is the min size of transversal set, where a subset of vertices is a transversal set if every edge in H contains ≥ 1 vertex from it.





tetrahedron

3-unif complete graph on 4-vertices

$\tau(C_5) = 3$ $\qquad$ $\tau(K_4^3) = 2$

- If the convex sets in F are polytope, then we can think of F as a hypergraph where a set of pts form an edge if their convex hull generates a set in F.

$$(p,q)\text{-property} \Rightarrow \tau(F) \leq HD_d(p,q)$$

- Consider $q=2$, $(p,2)$-property

Let G be the intersection graph of F: $V(G) = F$, $C \sim C'$ if $C \cap C' \neq \phi$

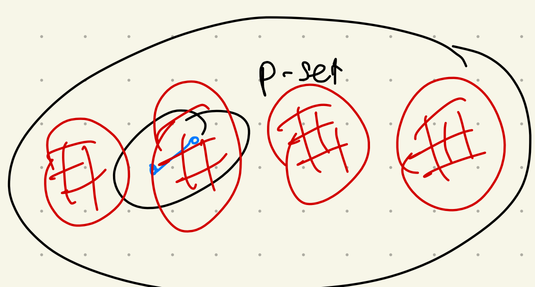
$\alpha(G) < p$

partition $F = F_1 \cup \dots \cup F_{\tau(F)}$

intersecting

$\Uparrow$

$\tau(F) = O_p(1) \Rightarrow$ partition $V(G)$ into $O_p(1)$ cliques



$\alpha(G) < p$

χ-boundedness

$$\Rightarrow \chi(\bar{G}) \leq \tau(F) = O_p(1) \Rightarrow \chi(\bar{G}) \leq f(\omega(\bar{G}))$$

This says that (hyper) graphs arise from geometry behaves nicely. (here means χ-boundedness)

<u>Observation</u> Suffices to assume $(p, d+1)$-property.

$(p \geq q > d)$ i.e. $HD_d(p,q) \leq HD_d(p, d+1)$

<u>First Attempt of proving (p,q)-thm</u>

- $(p, d+1)$-property $\xrightarrow{\text{double counting}}$ positive fraction of $(d+1)$-tuples of F intersecting
- Fractional Helly $\Longrightarrow$ $\Omega(n)$-size subfam. intersecting.
- iterate $\Rightarrow$ $O(\log n)$ upper bound on $\tau(F)$

<u>Idea</u> consider a blowup/weighted version.

Strategy of pf of (p,q)-thm

<u>Show</u> $\forall$ F w/ (p,q)-property is ε-chubby w.r.t. some set X.

Then (p,q)-thm is implied by weak ε-net thm.

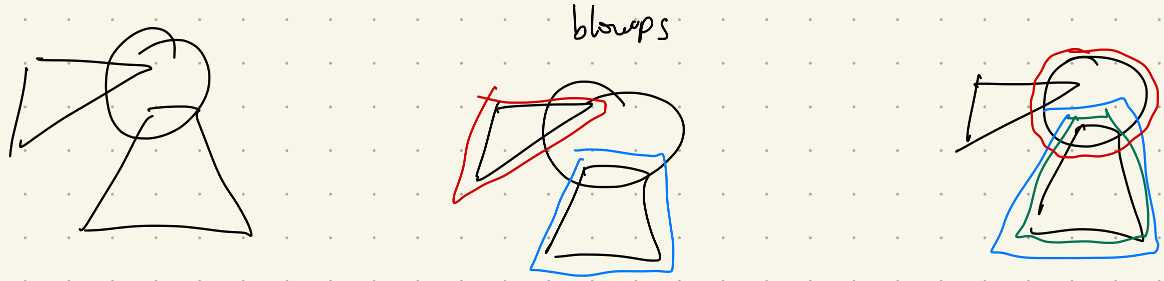
<u>Goal</u>: To construct a set X s.t. F is ε-chubby w.r.t. X i.e. $\forall\, C \in F$, $|C \cap X| \geq \varepsilon |X|$.

§ blowups of F

A blowup of F is a (multiset) fam. obtained by blowing up each members of F to arbitrary # copies.

F

blowups



- Key observation needed for the pf.

(p,q)-property is "hereditary" in the following sense:

Fact: If a finite fam. of sets in $\mathbb{R}^d$ has $(p, d+1)$-property $\Rightarrow$ any of its blowup F^* has

$$\Big(\overset{(p',\, d+1)}{(p-1)d+1},\ d+1\Big)\text{-property}$$

pf: $\forall$ p'-tuple in F^* contains
$$\begin{cases} \text{either one set } C \in F \ (d+1) \text{ times} \\ \text{OR } p \text{ distinct sets in } F \end{cases}$$