Lecture 16

Recall: Weak ε-net

Intuition: Large sets are easier to hit (large fish easier to catch)

- $X \subseteq \mathbb{R}^d$, define $F_\varepsilon = \{ \operatorname{conv} Y : Y \subseteq X, |Y| \geq \varepsilon |X| \}$

Another way to state weak ε-net thm:

Defn: A convex set C is ε-chubby w.r.t. X if $|C \cap X| \geq \varepsilon |X|$

A finite fam. of convex sets F is ε-chubby w.r.t. X if all sets in F is ε-chubby w.r.t. X.

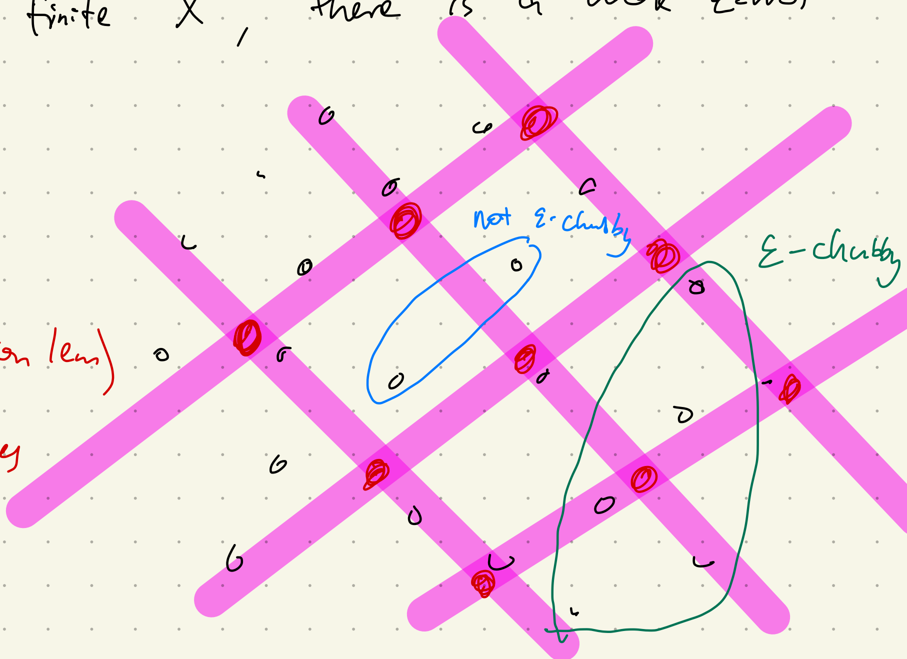
Defn. A weak ε-net for X is a set that pierces all sets in a finite fam F which is ε-chubby w.r.t. X.

Thm (weak ε-net thm) $\forall 0 < \varepsilon < 1$, $d \in \mathbb{N}$,

$\Rightarrow \exists f(\varepsilon, d)$ s.t. $\forall$ finite X, there is a weak ε-net of size $\leq f(\varepsilon, d)$.

Idea: greedy alg, repeatedly choosing a deepest pt (selection lem) & keep track of the X-simplices eliminated during the process.

$\tilde{O}(\varepsilon^{\frac{1}{d+1}})$



§ Better bound for planar case : $f(\varepsilon, 2) \le 7/\varepsilon^2$.

Defn. Given $X \subseteq \mathbb{R}^2$, n pts in general position, for $k \ge 3$,

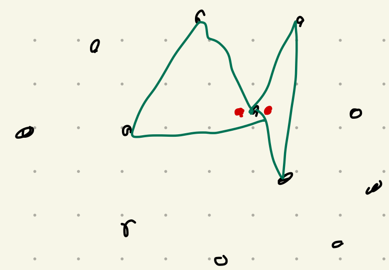
let $G_k = \{ \operatorname{conv} Y : Y \in \binom{X}{k} \}$.

defn. $g(n,k)$ = min # pts needed to piece all G_k for any given X.

$k = \varepsilon n$

Goal: $g(n,k) \le 7\left(\frac{n}{k}\right)^2 \implies f(\varepsilon, 2) \le 7/\varepsilon^2$. ☺

Pf by induction:



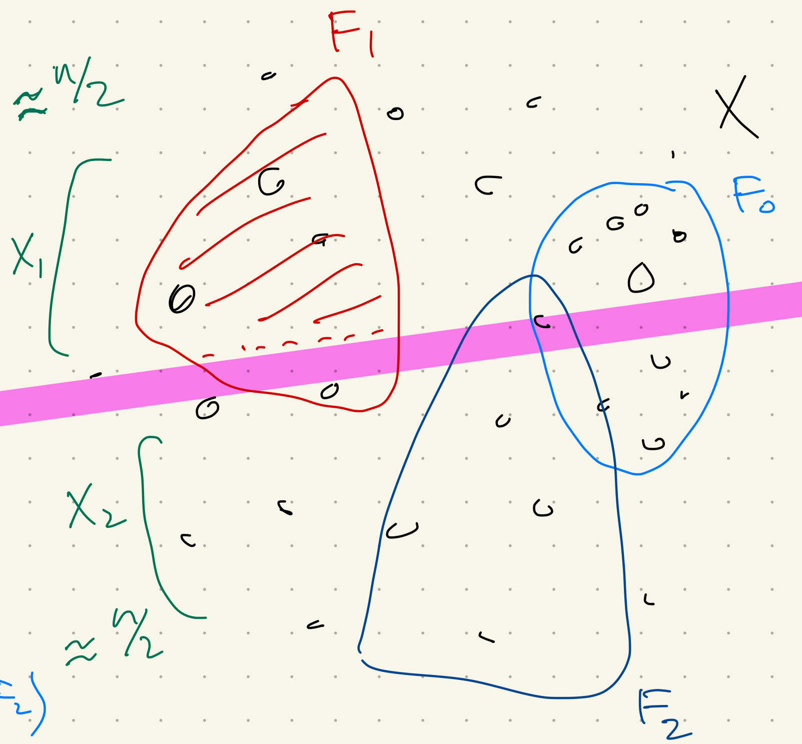
base case : $g(n,3) \le 2n-5$

(place two pts near each pt in X)

For $3 < k < 6$, checked by hand similarly.

Now assume $\overset{|X|=}{n} \ge k \ge 6$.

Classify $G_k : \operatorname{conv} Y, Y \in \binom{X}{k}$ as follows,



$F_0 = \{ \operatorname{conv} Y : Y \in \binom{X}{k}, |Y \cap X_i| > k/6 \}$

$F_1 = \{$ —— " —— $, |Y \cap X_1| \ge 5k/6 \}$

$F_2 = \{$ —— " —— $, |Y \cap X_2| \ge 5k/6 \}$

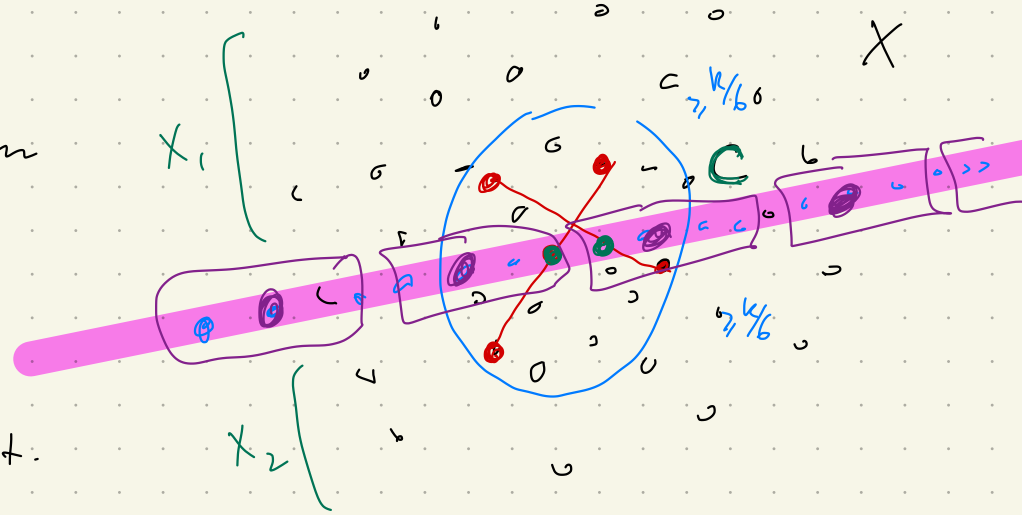
By induction hypo, F_1 (and also F_2) can be pierced by $\le g(n/2, 5k/6)$ pts

Fix a $C \in F_0$

pick a pair of pts from

$(C\cap X_1 , C\cap X_2)$

draw a line $\Rightarrow$ intersect the halving line at a pt.



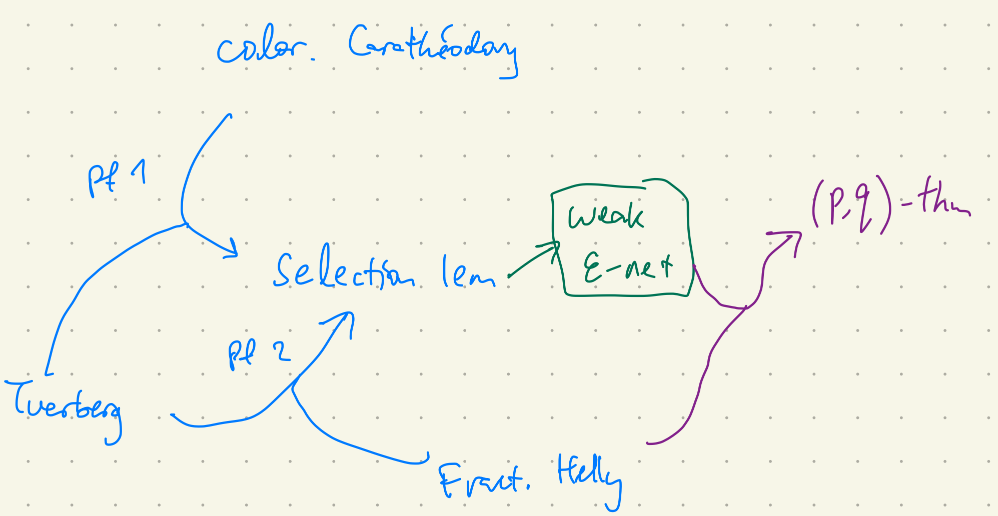
$\Rightarrow$ # such intersecting pts $= |C\cap X_1|\cdot|C\cap X_2| \geq \frac{k}{6}\cdot\frac{5k}{6} = \frac{5k^2}{36}$

total # of such intersecting $\leq \frac{n}{2}\cdot\frac{n}{2} = \frac{n^2}{4}$

we can cut them into segments of $\frac{5k^2}{36}$ pts each and take one pt from each segment.

$\Rightarrow$ pierce all sets in F_0

- $$g(n,k) \leq \underset{F_0}{\frac{n^2/4}{5k^2/36}} + \underbrace{2\,g(n/2, 5k/6)}_{F_1 \cup F_2} \cdots \leq 7\left(\frac{n}{k}\right)^2$$ ∎

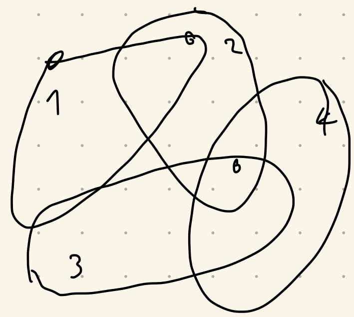


• Consider F a finite fam. of convex sets in $\mathbb{R}^2$

Helly: every 3 of them intersecting $\Longrightarrow$ all of F can be pierced by one point.

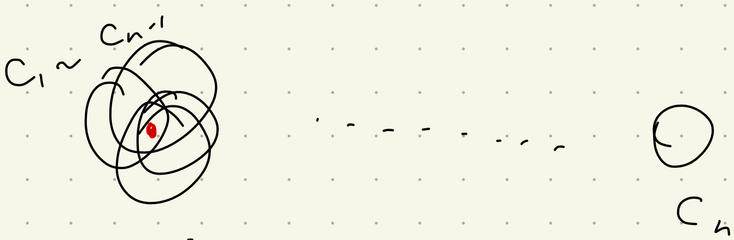
What if now we have the condition that among every 4 of them, 3 sets are intersecting

then we can no longer pierce the whole fam. by a single pt.



1, 2, 4 not intersecting
2, 3, 4 intersecting.

E.g. F =: union of $(n-1)$ sets that are intersecting & 1 set far away from all these $(n-1)$ sets.



$(n-1)$ int.

Q: Can we at least use bdd # pts to pierce all sets in F w/ such property ((4,3)-property)

This (4,3)-case is the simplest non-trivial case of the (p,q)-thm proved by Alon - Kleitman.

Defn. A family F of sets has (p,q)-property if every p-tuple in F contains a q-tuple that are intersecting.

Hadwiger-Debrunner conj

Thm (Alon-Kleitman (p,q)-thm)

$\forall\, p \geq q > d \geq 1$, $\Rightarrow \exists$ $HD_d(p,q)$ s.t. TFH.

If F is a finite fam. of convex sets in $\mathbb{R}^d$ w/

(p,q)-property $\Longrightarrow$ then members of F can be pierced by $\leq HD_d(p,q)$ pts.

Rmk : Helly : $(d+1, d+1)$-property $\Rightarrow$ pierced by one pt.

$HD_d(d+1, d+1) = 1$.