Lecture 9

Fractional Helly $\xleftarrow{\text{pf 2}}$ color. Helly

pf 1: Helly $\to$ Fractional Helly

Helly $\to$ color. Helly

Thm [Fractional Helly, Katchalski – Liu 79]

$\forall d \geq 1$, $\alpha > 0$, $\exists \beta = \beta(d, \alpha) > 0$ s.t. T.F.H.

Let $\mathcal{F} = \{F_1, \dots, F_n\}$ collection of convex sets in $\mathbb{R}^d$, $n \geq d+1$

If $\geq \alpha \binom{n}{d+1}$ of the $(d+1)$-tuples in $\mathcal{F}$ are intersecting

$\Longrightarrow$ then $\geq \beta n$ sets in $\mathcal{F}$ are intersecting

Rmk. • Optimal $\beta = 1 - (1-\alpha)^{\frac{1}{d+1}}$. (Notice $\beta \to 1$ as $\alpha \to 1$)

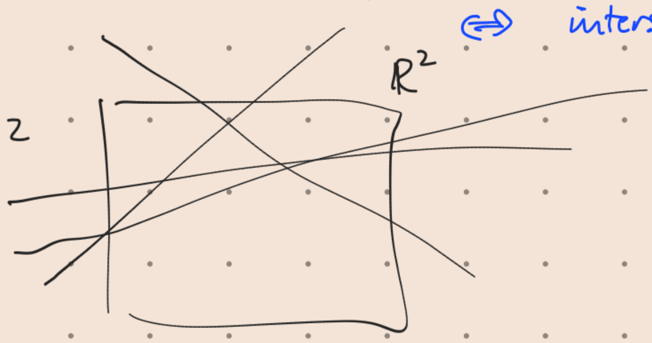
tight example : $\underbrace{(1-\beta)n \text{ hyperplanes}}_{\text{in general position}}$ $\cup$ βn copies of $\mathbb{R}^d$

meaning any $(d+1)$ of them have empty intersection

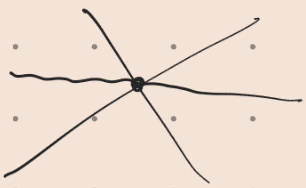
i.e. if $G = G^{(d+1)}$ $(d+1)$-unif intersection graph of $\mathcal{F}$

$(d+1)$-tuple form an edge $\Leftrightarrow$ intersecting

• $d=2$



$d=2$



$\bigcap_{i \in [d+1]} H_i \neq \emptyset$

NOT general position

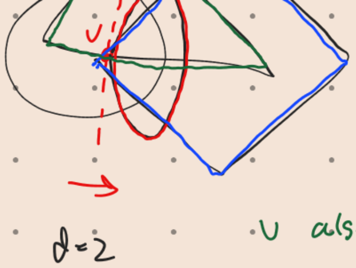
1st pf of fractional Helly (Helly $\Rightarrow$ fract. Helly)

• Will show for $\beta = \frac{\alpha}{d+1}$

Consequence of Helly: Lexicographic min. of intersection of $F_1, \dots, F_n$ is realised by $\leq d$ of them.

Lem. $\mathcal{F} = \{F_1, \dots, F_n\}$ intersecting

a unique lexi. min v. $\Longrightarrow$ v is the lexi. min of intersection of $\leq d$ of them.



v also min and

Pf: • $C = \{x \in \mathbb{R}^d : x <_{\text{lexi.}} v\}$ convex set

By defn. of v,

$C \cap (\cap \mathcal{F}) = \phi \cdots (*)$

• Apply Helly to $\mathcal{F} \cup \{C\}$, $(*) \Rightarrow \exists\, (d+1)$-tuple T of $\mathcal{F} \cup \{C\}$ with empty intersection. and $C \in T$.

i.e. $\mathcal{F}' := T - \{C\}$ is a d-tuple in $\mathcal{F}$.

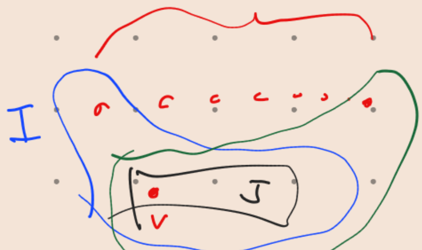
$\cap T = C \cap (\cap \mathcal{F}') = \phi \Rightarrow$ lexi. min of $\cap \mathcal{F}'$ is not in C, which has to be v. ∎

Pf: • $\forall$ intersecting $I \in \binom{[n]}{d+1} \xRightarrow{\text{lem}} J \in \binom{[n]}{d}$ s.t.

lexi. min I = lexi. min. J

by averaging,

• $\exists J \in \binom{[n]}{d}$ which is responsible for $\geq \boxed{\dfrac{\alpha\binom{n}{d+1}}{\binom{n}{d}}}$ $=$ ☺ intersecting $(d+1)$-tuples.



Let v = lexi. min. J, then

v is in $\geq \underset{\underset{J}{\uparrow}}{d} + ☺ \geq \dfrac{\alpha}{d+1} n$. ∎

β

2^{nd} pf

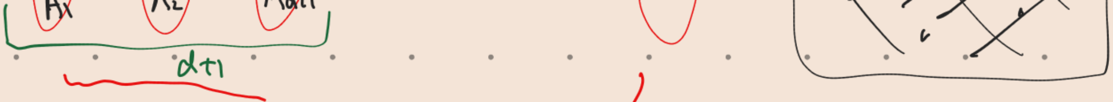
(M. Kim , $\beta := \max\left\{\dfrac{\alpha}{d+1}, 1-(d+1)(1-\alpha)^{\frac{1}{d+1}}\right\} \to 1$ as $\alpha \to 1$)

• Let $G = G^{(d+1)}$ be the intersection hypergraph of $(d+1)$-tuples in $\mathcal{F}$.

Will show contrapositive: max. intersecting subfam of $\mathcal{F}$ size $\leq \beta n \Rightarrow \leq \alpha\binom{n}{d+1}$ $(d+1)$-tuples in $\mathcal{F}$ are intersecting

• Take a maximal matching in G (non edges)

$|V \setminus M| \leq \beta n$



$$|M| \geq \frac{(1-\beta)n}{d+1}$$

- Note first that $V \setminus M$ is a complete (d+1)-graph by maximality of M.
- By Helly $\Rightarrow$ $V \setminus M^{\in \mathcal{F}}$ is intereseting $\Rightarrow$ $|V \setminus M| \leq \beta n$

$$\Rightarrow \quad |M| \geq \frac{(1-\beta)n}{d+1}$$

- $\underline{\text{NTS}}$ $\leq \alpha \binom{n}{d+1}$ intersecting (d+1)-tuples $\Longleftrightarrow$ $\overline{G}$ has edge-density $\geq (1-\alpha)$

  $\Updownarrow$

  G has edge density $\leq \alpha$

- For any $(d+1)$-tuple in M: $A_1, \dots, A_{d+1} \in \overline{G}$ $\Leftrightarrow$ $\bigcap A_i = \phi$

  by color. Helly $\Rightarrow$ $\exists$ transversal

  $a_i \in A_i$ s.t. $\bigcap_{i \in [d+1]} a_i = \phi$

  $\Rightarrow \{a_1, \dots, a_{d+1}\} \in \overline{G}$.

- Thus $e(\overline{G}) \geq \binom{|M|}{d+1} = \binom{\frac{(1-\beta)n}{d+1}}{d+1} \gtrsim \underbrace{\left(\frac{1-\beta}{d+1}\right)^{d+1}}_{\substack{\shortparallel \\ 1-\alpha}} \binom{n}{d+1}$ ▨

- Carathéodory: $X \subseteq \mathbb{R}^d$, $\operatorname{conv} X \ni p$ covered by X-simplices. $p \in \operatorname{conv} X'$, $|X'| \leq d+1$

$\underline{\text{Def}}$: conv hull of $(d+1)$ pts in X is a X-simplex.

$\underline{Q}$: Is there a pt covered by many X-simplices?

Point selection lem. says that the answer to the above question is a strong YES: $\exists$ pt covered by positive fraction of X-simplices.

k-simplices.