Lecture 15

· Announcement : No class next week.

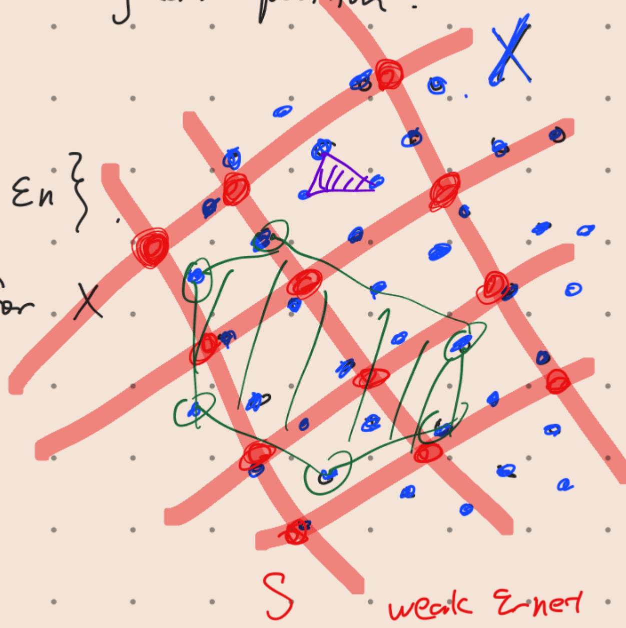
§ Weak ε-net

Fish $\Leftrightarrow$ polytope
convex body

Def: Let $X \subseteq \mathbb{R}^d$ be a set of n pts in general position.
For $\varepsilon > 0$, define

$$\mathcal{F}_\varepsilon = \{ \operatorname{conv} Y : Y \subseteq X, \ |Y| \geq \varepsilon n \}.$$

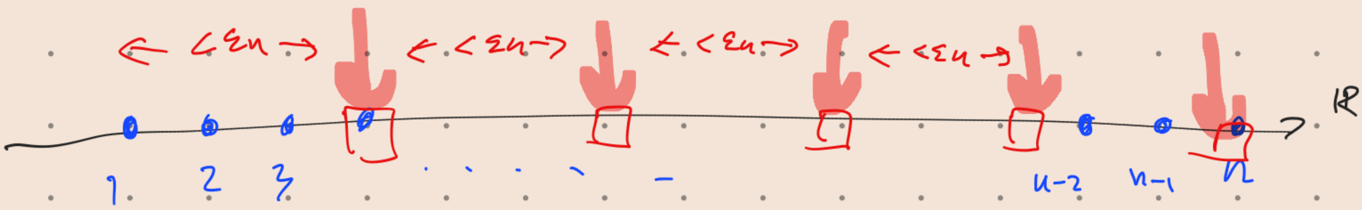
A finite set $S \subseteq \mathbb{R}^d$ is a weak ε-net for X
if $S \cap F \neq \emptyset \quad \forall F \in \mathcal{F}_\varepsilon$.



Rmk · Note that pts in the weak ε-net are not necessarily from X.

· Informally, weak ε-net only 'captures' the "chubby fish".

Ex $\mathbb{R}^1$, $X = \{1, 2, 3, \dots, n\}$

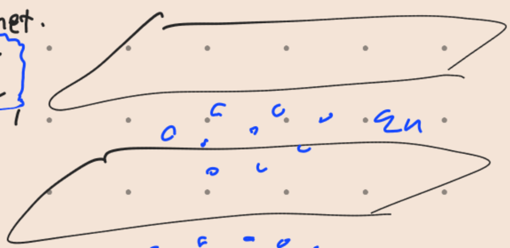


· Here in this example, the chubby fish are the intervals containing $\geq \varepsilon n$ pts in X.

· Take $\{ \lfloor \varepsilon n \rfloor, 2 \cdot \lfloor \varepsilon n \rfloor, \dots, t \lfloor \varepsilon n \rfloor \}$ where $t = \lfloor \frac{1}{\varepsilon} \rfloor$

$\Rightarrow$ weak ε-net of size $\leq 1/\varepsilon$.

Rmk: Cannot do better. In $\mathbb{R}^d$, weak ε-net $\boxed{\geq 1/\varepsilon}$
Take $1/\varepsilon + 1$ parallel hyperplanes in $\mathbb{R}^d$
and place εn pts between two consecutive



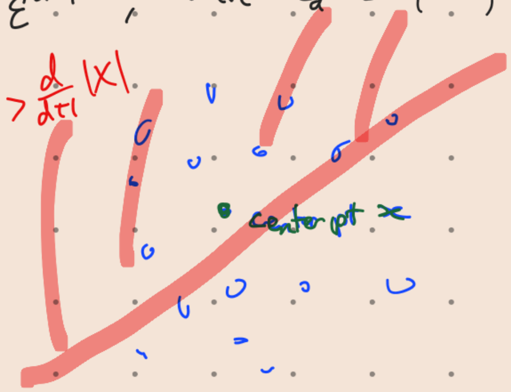
ones $\Rightarrow$ $\geq 1/\varepsilon$ pts in any weak ε-net. as we need ≥ 1 pts between any two consecutive hyperplane.

Thm: (weak ε-net thm) $\forall$ finite $X \subseteq \mathbb{R}^d$ and $\forall\, 0 < \varepsilon < 1$,

$\Rightarrow$ $\exists$ a weak ε-net of size $\leq c_d \frac{1}{\varepsilon^{d+1}}$, where $c_d = 2^{d+1}(d+1)^d$.

Rmk: • Connection to Helly: Center pt

$\forall\, \varepsilon > \frac{d}{d+1}$, Helly thm $\Rightarrow$ there is a weak ε-net of size 1.



• Best lower bound [Bukh - Matoušek - Nivasch] $\geq c_d \frac{1}{\varepsilon}\left(\log \frac{1}{\varepsilon}\right)^{d-1}$

Upper bound • Matoušek - Wagner $\leq \tilde{O}\left(\frac{1}{\varepsilon^d}\right)$

• Rubin $\leq O\left(\varepsilon^{-d-1/2-o(1)}\right)$.

Idea:
- each chubby fish contains many X-simplices $\left(\geq \binom{\varepsilon n}{d+1}\right)$

By selection lem
- deepest pt inside pierces many X-simplices $\left(\geq c_d' \binom{\varepsilon n}{d+1}\right)$
- so after $\leq \frac{\binom{n}{d+1}}{c_d'\binom{\varepsilon n}{d+1}} \leq c_d \frac{1}{\varepsilon^{d+1}}$ rounds, we pierces all X-simplices

Pf: • May assume $\varepsilon \leq \frac{1}{2(d+1)}$ (since $\forall\, \varepsilon' > \varepsilon$, any weak ε-net is also an weak ε'-net)

• May assume $n \geq \frac{2(d+1)}{\varepsilon}$ (o.w. $c_d \frac{1}{\varepsilon^{d+1}} \geq n$, we can take the whole X to be a weak ε-net)

Construct a weak ε-net iteratively.

Initiate $S_0 = \phi$ and $\mathcal{H}_0 = \binom{X}{d+1}$

At i-th step, if ...

At i-th step, if $\exists F \in \mathcal{F}_\varepsilon = \{ \operatorname{conv} Y : Y \subseteq X, |Y| \geq \varepsilon n \}$.

s.t. $F \cap S_{i-1} = \phi$

then apply selection lem. to get a pt x_i piercing $\geq \frac{1}{(d+1)^d} \binom{|Y|}{d+1}$ many X-simplices in F.

Set $S_i = S_{i-1} \cup \{x_i\}$.

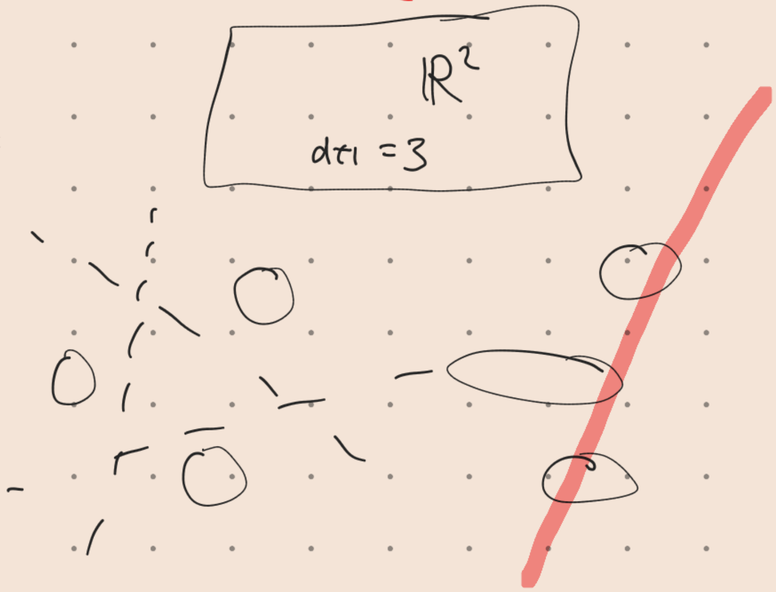
else terminate.

- By construct, the set S_t at the end is a weak ε-net.

and $t = |S_t| \leq \dfrac{\binom{n}{d+1}}{\frac{1}{(d+1)^d}\binom{\varepsilon n}{d+1}} \leq C_d \dfrac{1}{\varepsilon^{d+1}}$. ▨

$\varepsilon \leq \frac{1}{2(d+1)}$

$n \geq \frac{2(d+1)}{\varepsilon}$



Lem: Let $C_1, \dots, C_{d+1}$ be convex sets in $\mathbb{R}^d$, if $\forall$ nontrivial partition $[d+1] = A \cup B$,

$\bigcup_{i \in A} C_i$ and $\bigcup_{j \in B} C_j$ can be strictly separated by a hyperplane

$\Longrightarrow$ then there is no hyperplane simultaneously intersecting $C_1, \dots, C_{d+1}$.

Moreover, if $C_i = \operatorname{conv} X_i$ for some finite set $X_i \subseteq \mathbb{R}^d$

$\Rightarrow$ $(X_1, \dots, X_{d+1})$ has the same type transversal.

Idea [for same type lem] is to repeatedly apply Ham-sandwich to obtain large subsets that are well-separated.

For each partition $A \cup B = [d+1]$, we lose a factor ...

… $[d+1]$. … of $\leq 1/2$

from a hyperplane (from Ham-Sandwich).

We need to apply it for $\leq 2^{d+1}$ times. $\Rightarrow |Y_i| \geq 2^{-2^{d+1}} |X_i|$.

Tverberg

$\boxed{\text{Fractional Helly}}$ $\Rightarrow$ Selection lem $\rightarrow$ weak ε-net $\rightarrow$ (p,q)-thm