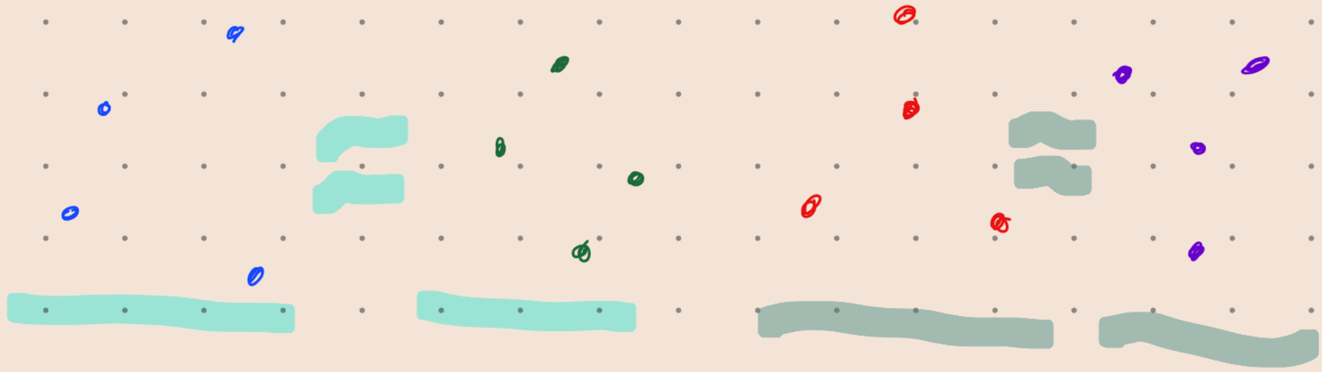
Lecture 14

§ Same type lem.

- Warm up in $\mathbb{R}^2$

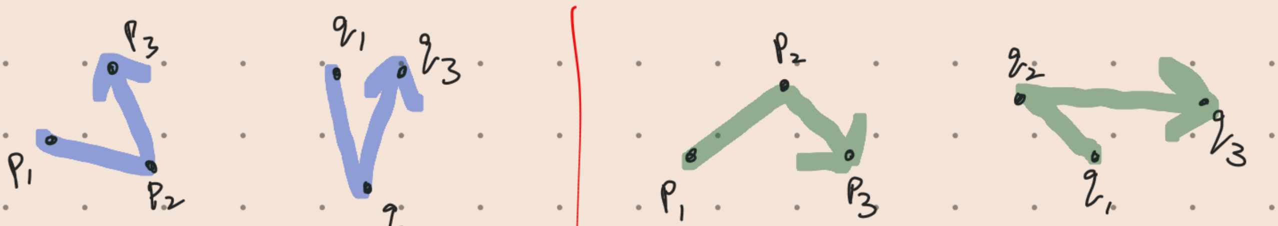
  Consider 4 pts in general position in $\mathbb{R}^2$.



Although there are ∞ many arrangements of 4 pts in g.p., there are only two types that are combinatorially different.

<u>Q</u>: What is a good notion to capture this comb. equivalence?

- Start with (ordered) triple of pts (p_1, p_2, p_3) & (q_1, q_2, q_3)



counterclockwise | clockwise

classify them by orientation

- generalise it to $\mathbb{R}^d$

  - ordered triple in $\mathbb{R}^2$

  - $\forall$ ordered d-tuple in $\mathbb{R}^d$ $(v_1, \cdots, v_d)$

    $\exists$ unique linear map $T = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_d \\ | & | & & | \end{pmatrix}$ $\Rightarrow Te_i = v_i$

    sending $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_i \longmapsto v_i$

d=3

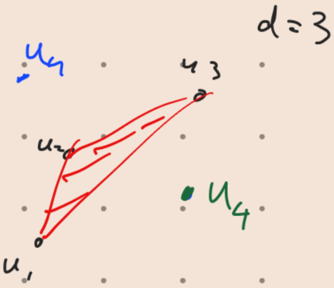
Def: Orientation of $(u_1, \cdots, u_d)$ is $\operatorname{sgn}(\det[\ \ ]) \in \{0, \pm 1\}$

The orientation of an ordered $(d+1)$-tuple $(u_1, \cdots, u_{d+1})$ in $\mathbb{R}^d$ is the same as that of the d-tuple $(u_2 - u_1, u_3 - u_1, \cdots, u_{d+1} - u_1)$.

Rmk. Equivalent way to define orientation of $(d+1)$-tuple in $\mathbb{R}^d$: $\operatorname{sgn}\left(\det\begin{pmatrix} u_1 & u_2 & \cdots & u_{d+1} \\ 1 & 1 & & 1 \end{pmatrix}\right)$.

Rmk: If $u_1, \ldots, u_{d+1}$ are affinely indep. (i.e. orientation $\neq 0$)

geometrically, the orientation of $(u_1, \cdots, u_{d+1})$ tells us which side of the hyperplane spanned by $u_1, \cdots, u_d$, the pt u_{d+1} lies in.

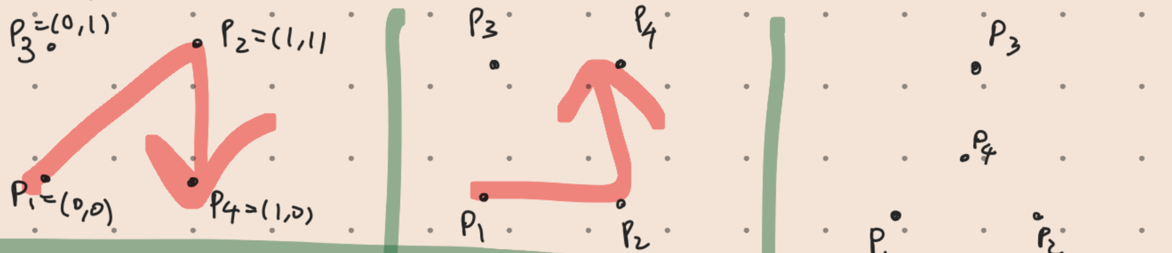


Def: The order type of an n-tuple $\mathbf{p} = (p_1, \cdots, p_n)$ in $\mathbb{R}^d$ ← bold font

is the funct. $\chi_{\mathbf{p}} : \binom{[n]}{d+1} \to \{0, \pm 1\}$.

Two tuples p & q have the same order type if $\chi_p = \chi_q$

Ex 4-tuples in $\mathbb{R}^2$



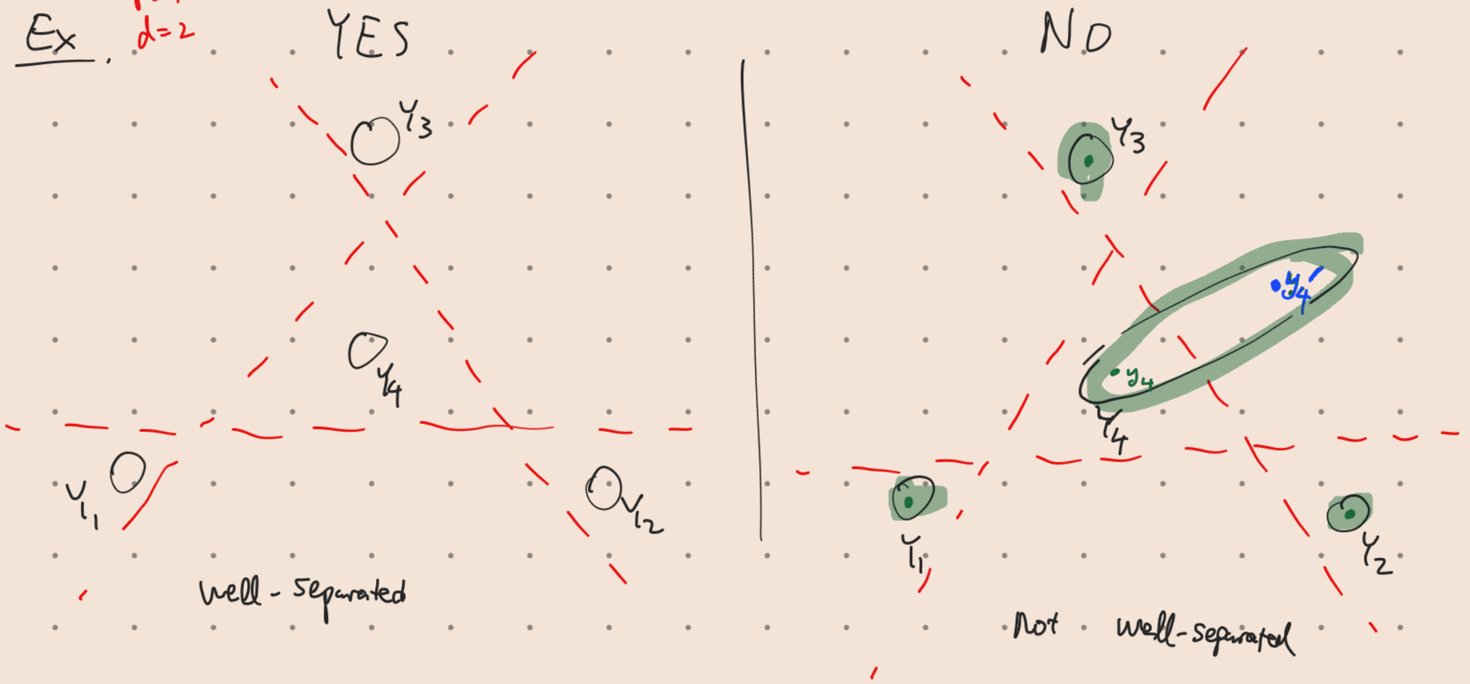
123	$\lvert p_2 - p_1, p_3 - p_1 \rvert = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$	1	1
124	-1	1	1
134	-1	-1	-1
234	1	-1	1

Det: (Same type transversal) We say an r-tuple of finite sets

transversals

$(Y_1, Y_2, \dots, Y_r)$ has the same type transversal if all $(y_1, \dots, y_r) \in (Y_1, \dots, Y_r)$ have the same type.

Ex. $r=4$, $d=2$

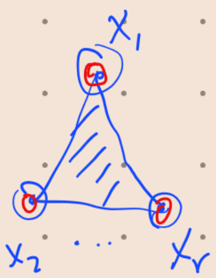


Using this notion, we can restate the Ramsey type "Same type lem" as follows.

Lem: $\forall$ $(X_1, \dots, X_r)$ in $\mathbb{R}^d$ (r-tuple of finite sets with $X_1 \cup \dots \cup X_r$ in general position) $\Rightarrow \exists\, Y_i \subseteq X_i$ w/ $\underline{|Y_i| = \Omega(|X_i|)}$ s.t. $(Y_1, \dots, Y_r)$ have the same type transversal.

Rmk: Qualitatively, this can be proved

(find $|Y_i| \to \infty$ as $|X_i| \to \infty$)



- # types of a transversal of $(X_1, \dots, X_r) \le k = 2^{\binom{r}{d+1}}$
- color transversals by their types. $\Rightarrow$ k-edge-colored complete r-partite r-unif hypergraph
- $|X_i| \overset{=n}{}$ large, r, d (hence k) are fixed

  $\Rightarrow$ $\geq$ one colour class is dense ($\geq 1/k$ fraction of edges)

By Erdős-Sim. Supersaturation $\Rightarrow \exists\, K^{(r)}_{\underbrace{t,\dots,t}_{r\text{-part}}}$ in this color, $t \to \infty$ as $n \to \infty$

Rmk: So the merit of same type lem is that it finds a huge subset Y_i ($t = \Omega(n)$).. Common phenomenon: Ramsey type results in geometric settings often have much better bounds.

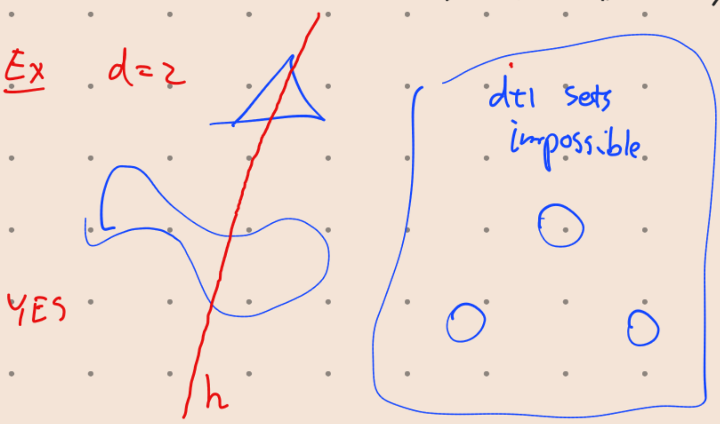
Idea (same type lem): iteratively apply Ham-Sandwich thm to obtained well-separated sets (which have the same type transversal).

Thm [Ham-Sandwich]

$\forall$ finite sets $A_1, \dots, A_d \subseteq \mathbb{R}^d \Rightarrow \exists$ a hyperplane bisecting all of them simultaneously.

Def: $\leq$ half of pts on each side of the hyperplane

Ex $d=2$



Def: A tuple of sets $(Y_1, \dots, Y_{d+1})$ on $\mathbb{R}^d$ are well-separated if $\forall$ partition $A \cup B = [d+1]$, $\left(\bigcup_{i \in A} \operatorname{conv} Y_i\right) \cap \left(\bigcup_{j \in B} \operatorname{conv} Y_j\right) = \phi$.

Lem: If $(Y_1, \dots, Y_{d+1})$ on $\mathbb{R}^d$ are well-separated, then they have the same type transversal.