Lecture 13

Homogeneous selection lem

Thm For $d \in \mathbb{N}$, $\exists\, c = c(d) > 0$ s.t. T.F.H.

Let $X \subseteq \mathbb{R}^d$ be a set of pts in general position w/ a partition

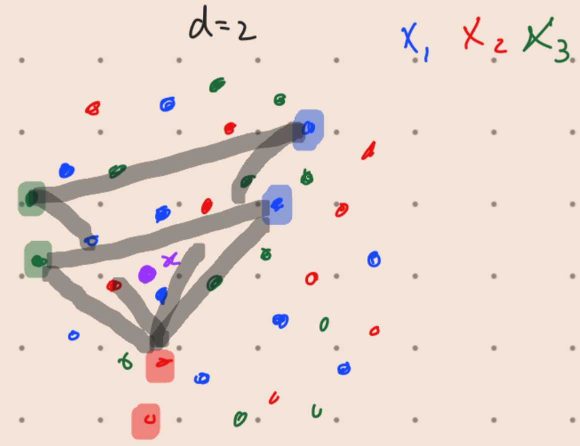
$X = X_1 \cup \cdots \cup X_{d+1}$, where $|X_i| = n$ for each $i \in [d+1]$.

$\Rightarrow \exists$ • $Y_i \subseteq X_i$ w/ $|Y_i| \geq c \cdot n$ for all $i \in [d+1]$

• and a pt $x \in \mathbb{R}^d$ s.t.

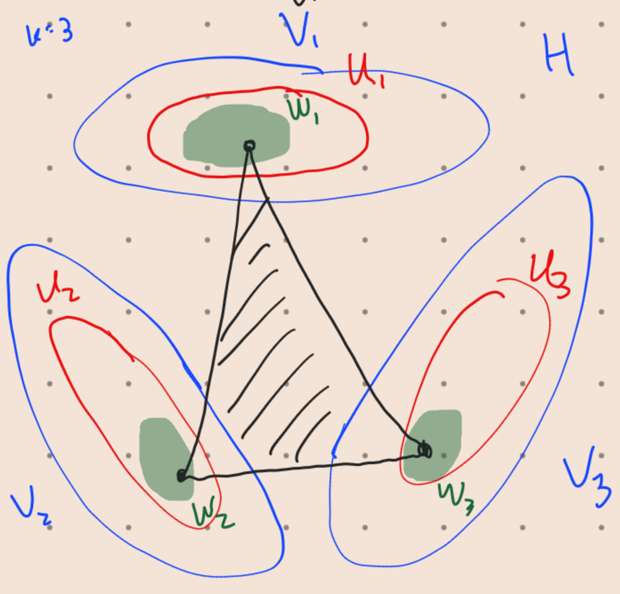
$x \in \operatorname{conv}\{y_1, \dots, y_{d+1}\}$ for all

$(y_1, \dots, y_{d+1}) \in Y_1 \times \cdots \times Y_{d+1}$.



tools

- 2^{nd} selection lem
- weak regularity
- same type lem.



Thm (Weak hypergraph regularity)

Let H be a k-partite k-unif. hypergraph with vertex classes $V_1, \dots, V_k$, each of size n. Suppose $\frac{e(H)}{|V_1| \cdots |V_k|} \geq \beta > 0$, $\varepsilon > 0$ and n suff. large.

Then $\exists\, U_i \subseteq V_i$ of size $|U_i| \geq \beta^{\frac{1}{\varepsilon^k}} n$ for each $i \in [k]$ s.t.

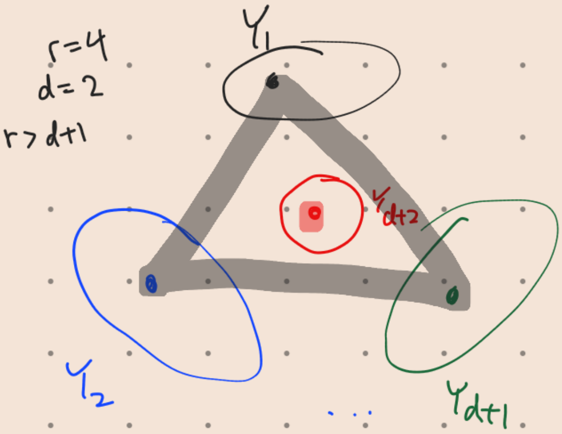
(i) $e_H[U_1, \dots, U_k] / |U_1| \cdots |U_k| \geq \beta$,

(ii) $\forall\, W_i \subseteq U_i$ w/ $|W_i| \geq \varepsilon |U_i|$

$e_H[W_1, \dots, W_k] \geq 1$.

Thm (Same type lem) $\forall\, r > d+1$, $\exists\, c = c(d, r) > 0$ s.t. T.F.H.

Let $X_1, \dots, X_r$ be disjoint sets of pts in $\mathbb{R}^d$ w/ $X_1 \cup \cdots \cup X_r$ in general position. $\Rightarrow \exists\, Y_i \subseteq X_i$ for each $i \in [r]$ w/ $|Y_i| \geq c|X_i|$



s.t. all $(y_1, \dots, y_r) \in Y_1 \times \dots \times Y_r$ have the same order type.

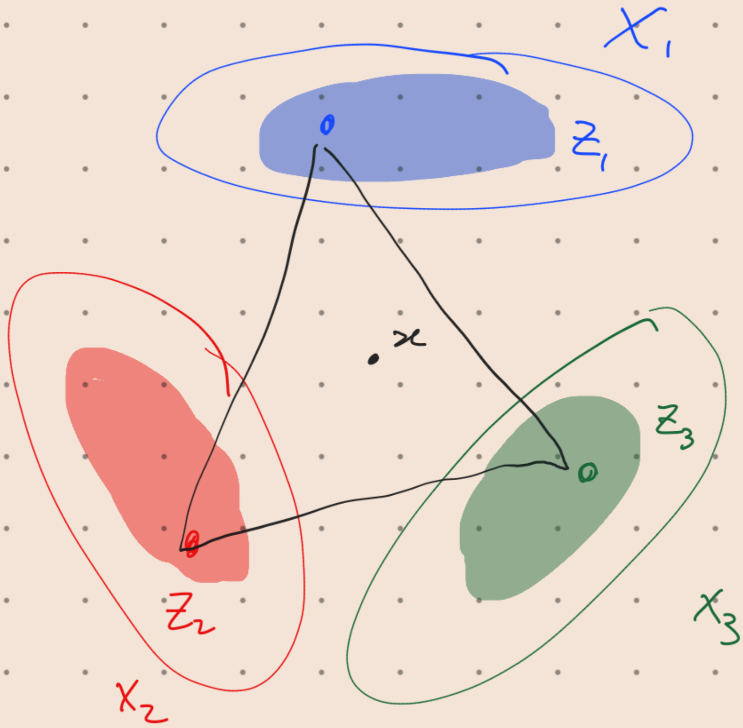
Rmk. For the pf of Homog. Selection lem, we use only the case $r = d+2$. For now, it is enough to understand $(y_1, \dots, y_{d+2})$ same type, meaning

if $y_{d+2} \in \operatorname{conv}\{y_1, \dots, y_{d+1}\}$ then $y'_{d+2} \in \operatorname{conv}\{y'_1, \dots, y'_{d+1}\}$ for all $(y'_1, \dots, y'_{d+2}) \in Y_1 \times \dots \times Y_{d+2}$.

Pf. (Homog. selection lem)

- Let $\mathcal{H}$ be the complete $(d+1)$-partite $(d+1)$-unif hypergraph on partite sets $X_1, \dots, X_{d+1}$.

  As $\mathcal{H}$ has positive edge density, by 2^{nd} selection lem $\Rightarrow \exists\, x \in \mathbb{R}^d$ piercing positive fraction of edges of $\mathcal{H}$ ($(X_1, \dots, X_{d+1})$-simplices)



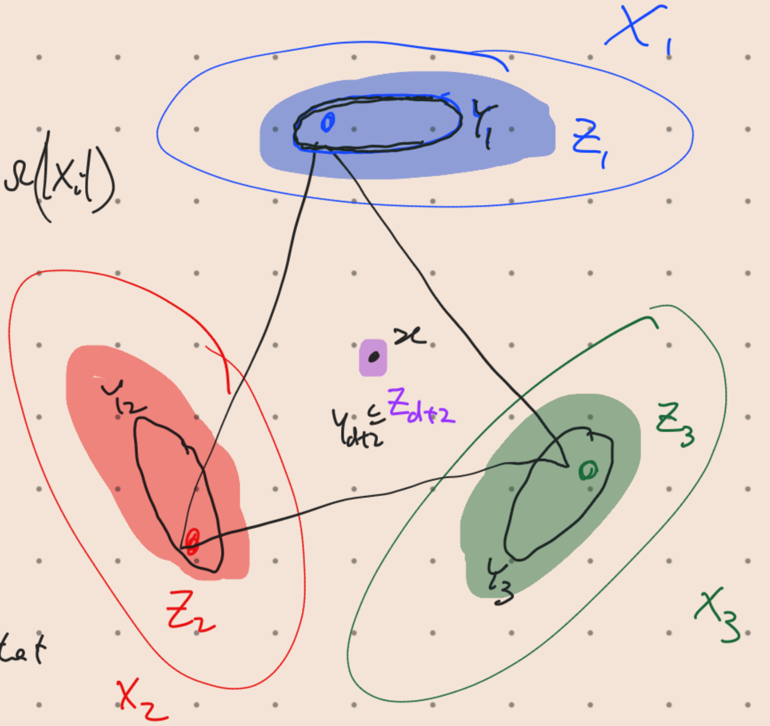
- Let $\mathcal{F} \subseteq \mathcal{H}$ be the subhyp. w/ all edges corresponding to rainbow X-simplices containing x. So $\mathcal{F}$ has positive edge density.

  Apply weak hyp. reg. lem on $\mathcal{F}$ $\Rightarrow Z_i \subseteq X_i$, $\begin{cases} \cdot\ |Z_i| = \Omega(|X_i|) \\ \cdot\ \text{any } Y'_i \subseteq Z_i \text{ w/ } |Y'_i| = \Omega(|Z_i|) \text{ induces} \geq \text{one edge in } \mathcal{F}. \end{cases}$

- Shall apply same type lem w/ $r = d+2$ on sets

$Z_1, \dots, Z_{d+1}, Z_{d+2}$, where

$Z_{d+2} = \{x, \dots, x\}$ with $|Z_{d+2}| = \Omega(|X_i|)$
multiset



$\Rightarrow Y_i \subseteq Z_i$, $i \in [d+2]$, $|Y_i| = \Omega(|Z_i|)$

s.t. all $Y_1 \times \dots \times Y_{d+2}$ transversals have the same type.

- By choices of Z_i and that

  $Y_i \subseteq Z_i$, $|Y_i| = \Omega(|Z_i|)$

  $\Rightarrow e_{\mathcal{F}}[Y_1, \dots, Y_{d+1}] \geq 1$

  This means that some $\operatorname{conv}\{y_1, \dots, y_{d+1}\} \ni x$.
  Y_{d+2}

  As all $Y_1 \times \dots \times Y_{d+2}$ transversals have the same type:

  $x \in$ conv hull of all $Y_1 \times \dots \times Y_{d+1}$ transversal. $\blacksquare$

$$X_i \xrightarrow[\text{hyp. reg}]{\text{weak}} Z_i \xrightarrow[\text{type}]{\text{same}} Y_i$$