Lecture 12

Recall

- 1^{st} selection lem: X: n pts in general position in $\mathbb{R}^d$ $\Longrightarrow$ $\exists$ a pt piercing a positive fraction of X-simplices.

Lem (2^{nd} Selection lem) For $d \in \mathbb{N}$, $\exists\ s(d) > 0$ s.t. T.F.H.

Let $\alpha\,\alpha \leq 1$ and $X \subseteq \mathbb{R}^d$ be a set of n pts in general position.

Let $\mathcal{F} \subseteq \binom{X}{d+1}$ be a family of $\alpha \binom{n}{d+1}$ X-simplices.

$\Longrightarrow$ Then $\exists$ a pt in $\mathbb{R}^d$ contained in $\geq s(d) \cdot \alpha^{s(d)} \binom{n}{d+1}$ many simplizes in $\mathcal{F}$.

- Erdős-Simonovits supersaturation + colorful Tverberg + Fractional Helly $\Longrightarrow$ 2^{nd} Selection lem.

Def: Given graphs H and G, the H-density in G is defined as

$$t(H; G) = \frac{\#\text{ copies of } H \text{ in } G}{\binom{|G|}{|H|}}.$$

Thm [Erdős-Simonovits Supersaturation]

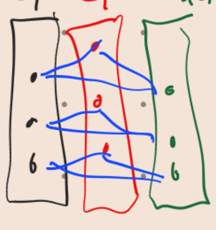
Let $\alpha > 0$ and $t \in \mathbb{N}$. Let $\mathcal{H}$ be a k-unif. hypergraph w/ edge-density α. $\Longrightarrow$ Then $t(K^{(k)}_{\underbrace{t,\dots,t}_{k \text{ parts}}}, \mathcal{H}) > c(k,t)\,\alpha^{t^k}$.

- We will use it for fixed α, k, t, while $|\mathcal{H}| \to \infty$. So this supersaturation says if edge-density is positive, then any of its (fixed) blowups has also positive density.

- Colorful Tverberg: $|C_i| \geq 2r+2$ $\forall i \in [d+1]$

$\Rightarrow$ $\exists$ disjoint rainbow sets

$X_1, \dots, X_r$ s.t.



$$\bigcap_{i\in[d+1]} \mathrm{conv}\, X_i \neq \emptyset$$

<u>Thought process</u>: Given $\mathcal{F} \subseteq \binom{X}{d+1}$ $|\mathcal{F}| = \alpha\binom{n}{d+1}$

- If we know that positive fraction of $(d+1)$-tuples in $\mathcal{F}$ are intersecting $\Rightarrow$ by fractional Helly $\exists$ pt piercing positive fraction of members of $\mathcal{F}$. ☺

- Think of $\mathcal{F}$ as a $(d+1)$-unif hypergraph

Suppose we have a huge clique $K_t^{(d+1)}$ in $\mathcal{F}$

$\xrightarrow[\text{Tverberg}]{t \text{ large}}$ $\exists$ disjoint $X_1, \dots, X_{d+1}$ w/ $\bigcap_{i\in[d+1]} \mathrm{conv}\, X_i \neq \emptyset$

So if we have positive $K_t^{(d+1)}$-density in $\mathcal{F}$, we are done.

Problem is that not even a single $K_t^{(d+1)}$ is guaranteed.

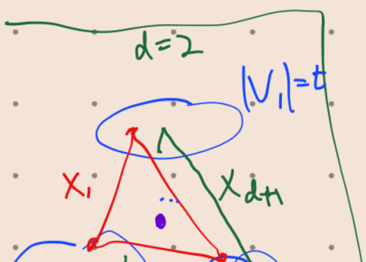
Consider eg $d=1$, $\mathcal{F} = K_{n/2, n/2}$. $\begin{cases} \Delta\text{-free} \\ \text{positive edge-density} \end{cases}$

This is where colorful Tverberg kicks in. While we are not guarantee any large clique by positive edge-density, we have many (positive fraction) of $\underbrace{K^{(d+1)}_{t,\dots,t}}_{d+1 \text{ parts}}$ due to Erdős–Simonovits supersaturation

And

$K_t^{(d+1)} \xrightarrow{\text{Tverberg}}$ intersecting $(d+1)$-tuple

$K^{(d+1)}_{t,\dots,t} \xrightarrow{\text{colorful Tverberg}}$ intersecting $(d+1)$-tuple

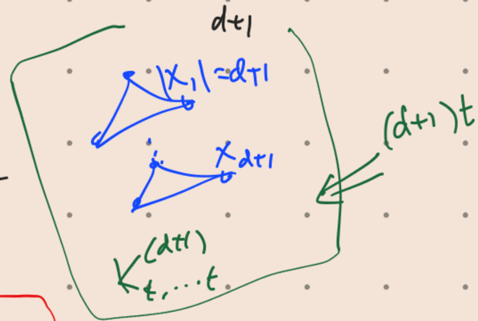


<u>Pf</u> (2nd selection) • View $\mathcal{F}$ as $(d+1)$-unif hypergraph w/ edge density $\alpha > 0$. By fractional Helly, it suffices to show that positive fraction of $\binom{\mathcal{F}}{d+1}$ are intersecting.

• Let $t = 2d+4$, then by colorful Tverberg, every copy of $K^{(d+1)}_{\underbrace{t,\dots,t}_{d+1 \text{ parts}}}$ yields an intersecting $(d+1)$-tuple in $\binom{\mathcal{F}}{d+1}$

• By Erdős–Sim., $\# K^{(d+1)}_{\underbrace{t,\dots,t}_{d+1}}$ in $\mathcal{F} \geq \overbrace{c_d \alpha^{t^{d+1}}}^{c'_{d,\alpha}} n^{(d+1)t}$, each one of which yields an intersecting $(d+1)$-tuples in $\mathcal{F}$.

• Note that each such intersecting $(d+1)$-tuple is contained in at most $n^{(d+1)t-(d+1)^2}$ copies of $K^{(d+1)}_{\underbrace{t,\dots,t}_{d+1}}$.



$\Longrightarrow$ # intersecting $(d+1)$-tuples in $\mathcal{F}$ is

$$\geq \frac{c'_{d,\alpha}\, n^{(d+1)t}}{n^{(d+1)t-(d+1)^2}} = c'_{d,\alpha} \cdot n^{(d+1)^2}$$

$$\geq \cdots \geq c''_{d,\alpha} \boxed{\binom{\binom{n}{d+1}}{d+1}}.$$

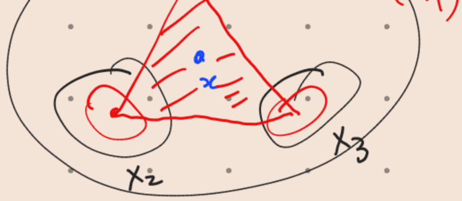
$$\geq c'''_{d,\alpha} \binom{|\mathcal{F}|}{d+1}.$$

∎

Homogeneous selection lemma

$\mathcal{F}$ = all X-simplices

$\mathcal{F} \subseteq X$-simplices
positive fraction



$x \in$ all Y_i-transversal.

Thm (Homogeneous selection, Pach)

For $d \in \mathbb{N}$, $\exists\, c(d) > 0$ s.t. T.F.H.

Let $X \subseteq \mathbb{R}^d$ be a set of pts in general position and $X = X_1 \cup \dots \cup X_{d+1}$ be a partition w./ $|X_i| = n$, $i \in [d+1]$.

$\Longrightarrow$ Then $\exists \begin{cases} Y_i \subseteq X_i, & i \in [d+1], \\ \text{a pt } x \in \mathbb{R}^d \end{cases}$ s.t. $|Y_i| \geq c(d) \cdot |X_i| \quad \forall i \in [d+1]$
and
$a \in \operatorname{conv}\{y_1, \dots, y_{d+1}\}$
for any transversal $y_i \in Y_i$, $i \in [d+1]$.