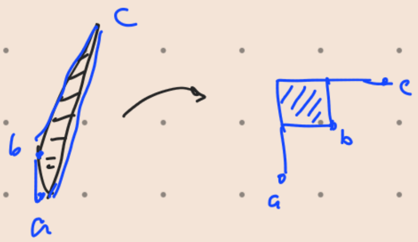
Lecture 11

Recap Boros Füredi : $\forall X \subseteq \mathbb{R}^2$ general position, $\exists p \in \mathbb{R}^2$ contained in $\geq \frac{2}{9}\binom{n}{3} + O(n^2)$

optimal by stretched grid / staircase convex set



Thm (1st point selection lem) $\forall d \in \mathbb{N}$ $\exists c(d) > 0$ s.t. T.F.H.

$\forall X \subseteq \mathbb{R}^d$ n pts in general position $\Longrightarrow$ $\exists$ a pt in $\geq c(d)\binom{n}{d+1} - O(n^d)$ simplices spanned by X.

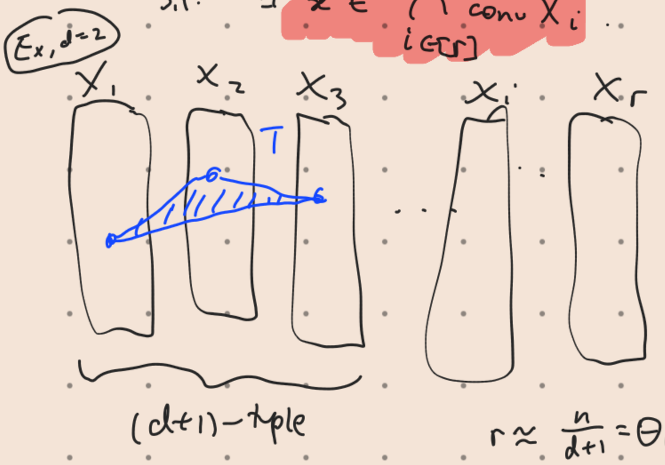
$n \geq d+1$

1st **Pf** (Tverberg + colorful Cara.)

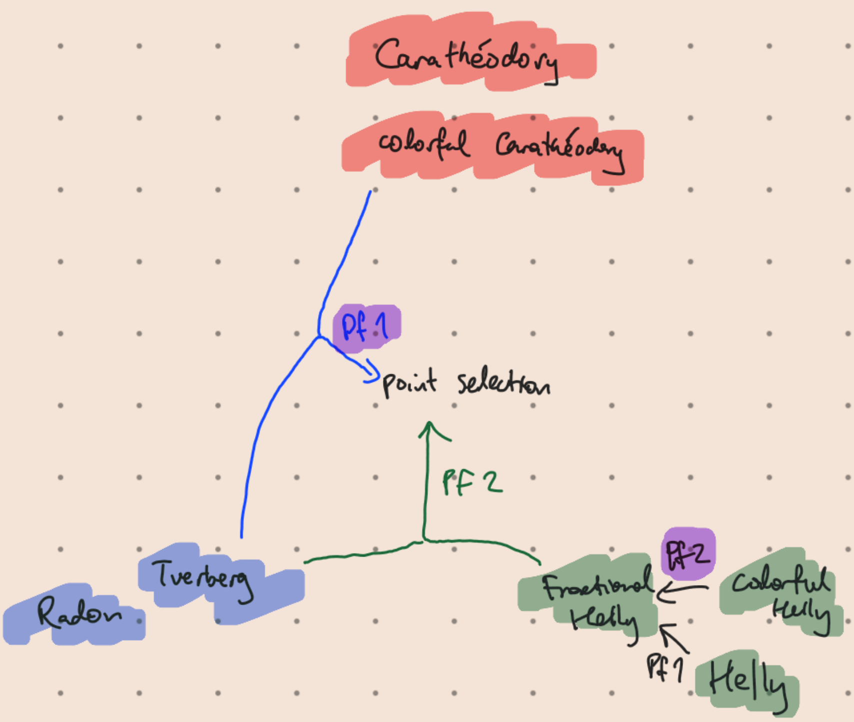
- Write $n = (r-1)(d+1) + k$, $1 \leq k \leq d+1$

Tverberg $\Rightarrow$ $\exists$ $X = X_1 \cup \cdots \cup X_r$

s.t. $\exists$ $x \in \bigcap_{i \in [r]} \operatorname{conv} X_i$.



$r \approx \frac{n}{d+1} = \Theta(n)$



- For any $(d+1)$-tuple of $X_{i_1}, \ldots, X_{i_{d+1}}$, as $x \in \operatorname{conv} X_i$ $\forall i$

By colorful Carathéodory, $\exists$ transversal T s.t. $x \in \operatorname{conv} T$

X-simplex

$\Longrightarrow$ # X-simplices containing $x \geq \binom{r}{d+1} \approx \binom{n/(d+1)}{d+1} \gtrsim \frac{1}{(d+1)^{d+1}}\binom{n}{d+1}$. ▨

2nd **Pf** (Tverberg + Fractional Helly)

- Consider $\mathcal{F} = \{X\text{-simplices}\}$

a pt in positive fraction of X-simplices $\Longleftrightarrow$ Finding positive fraction of $\mathcal{F}$ intersecting: $\mathcal{F}' \subseteq \mathcal{F}$ s.t. $|\mathcal{F}'| = \Omega(|\mathcal{F}|)$, $\bigcap \mathcal{F}' \neq \phi$.

- By Fractional Helly, to show $\longrightarrow$ it suffices to prove that positive fraction of $(d+1)$-tuples of $\mathcal{F}$ are intersecting.

- $\forall\, Y \subseteq X$, $|Y| = (d+1)^2 > \overset{(r-1)}{d}(d+1)+1$, by Tverberg

  $\exists$ $(d+1)$-partition: $Y = Y_1 \cup \cdots \cup Y_{d+1}$ s.t. $\bigcap_{i \in [d+1]} \operatorname{conv} Y_i \neq \phi$

  By Carathéodory, may assume $|Y_i| \leq d+1$.

  So each Y_i is a X-simplex.

  That is, $\forall\, (d+1)^2$ pts in X, Tverberg gives an intersecting $(d+1)$-tuples in $\mathcal{F}$.

  $\Rightarrow$ # intersecting $(d+1)$-tuples in $\mathcal{F}$ $\geq \binom{n}{(d+1)^2} \geq c(d) \binom{n}{d+1}$. ▨

Rmk 1. The pt in positive fraction of X-simplices guaranteed might not be a pt in X.

2. $\dfrac{2d}{(d+1)!\,(d+1)} \overset{\text{Gromov}}{\leq} c(d) \leq \dfrac{(d+1)!}{(d+1)^{d+1}}$ Bukh – Matoušek – Nivasch.

conj to be tight. $d=2 \Rightarrow 2/9$

§ Upper bound on covering #.

covering # : # X-simplices $\ni p$

- 1st point lem studies $\min_{|X|=n} \max_{p \in \mathbb{R}^d} m_X(p)$

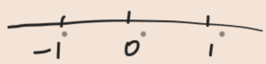
What about $\max_{|X|=n} \max_{p \in \mathbb{R}^d} m_X(p)$?

Thm $\forall\, X \subseteq \mathbb{R}^d$ n pts, $n \gg d$, in general position then $\Rightarrow \forall\, p \in \mathbb{R}^d$, $\quad m_X(p) \le (1+o(1)) \frac{1}{2^d} \binom{n}{d+1}$.

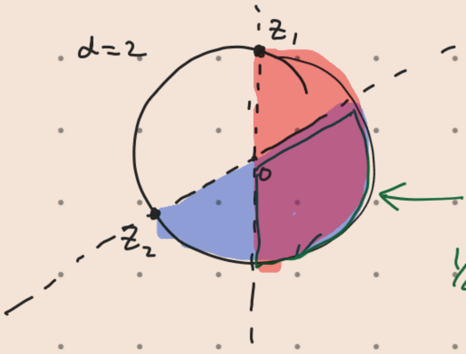
- The pf uses the upper bound theorem by McMullen.
- $1/2^d$ is best possible, implied by the following lem.

__Lem__: Let $z_1, \dots, z_{d+1}$ be uniform indep pts from the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ $\Longrightarrow$ $\Pr(0 \in \operatorname{conv}\{z_1, \dots, z_{d+1}\}) = 1/2^d$.

d=1



$0 \in$ conv hull $\Leftrightarrow$ 2 pts on diff side



$\leftarrow z_3 \Leftrightarrow 0 \in$ conv hull

$1/4$

__Thm__ (2^{nd} point selection lem) $\forall\, d \in \mathbb{N}$, $\exists\, s(d) > 0$ s.t. T.F.H.

Let $X \subseteq \mathbb{R}^d$ be n pts in general position and $\mathcal{F}$ be a family of $\alpha \binom{n}{d+1}$ X-simplices, $\alpha > 0$.

$\Longrightarrow$ Then $\exists$ a pt in $\mathbb{R}^d$ contained in $\geq \alpha^{s(d)} \binom{n}{d+1}$ of the simplices in $\mathcal{F}$.

Tools:
- Erdős - Simonovits
- Colorful Tverberg
- Fractional Helly

Turán density of H

$n^{2-c_H} \leftarrow ex(n,H)$

$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = 0$

For bip H,

__Thm__ (Erdős - Simonovits) [Complete bipartite graphs have 0 Turán density]

Let $\mathcal{H}$ be an n-vertex k-uniform hypergraph w/ $\alpha\binom{n}{k}$ edges

where $\alpha \geq C n^{-1/t^{k-1}}$, then $\mathcal{H}$ contains at least

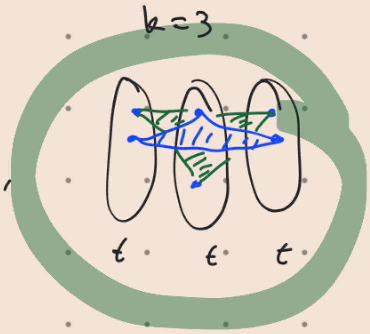
$\Omega(\alpha^{t^k} n^{kt})$ copies of $K^{(k)}_{\underbrace{t,\dots,t}_{k \text{ part.}}}$.

<u>Rmk</u> Consider $\alpha > 0$ fixed constant,

note #vxs in $K^{(k)}_{t,..,t}$ is kt. So E-S.

Says that for k-unif hypergraphs w/ positive edge density,

the complete k-partite $K^{(k)}_{t,\dots,t}$ has positive density.



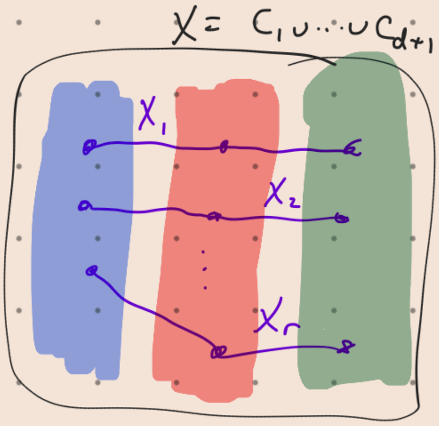
<u>Thm</u> (Colorful Tverberg)

Let $C_1, \dots, C_{d+1} \subseteq \mathbb{R}^d$ w/ $|C_i| = 2r+2$.

Then $\exists$ disjoint transversals $X_1, \dots, X_r$

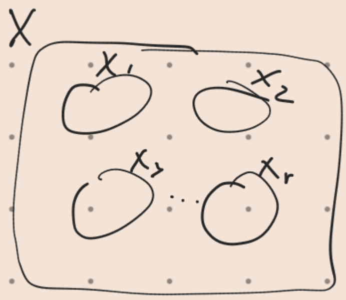
(i.e. $\forall i,j$ $|X_i \cap C_j| = 1$) s.t.

$$\bigcap_{i \in [r]} \operatorname{conv} X_i \neq \phi$$



Recall Tverberg

$|X| = (r-1)(d+1)+1$



$$\bigcap_{i \in [r]} \operatorname{conv} X_i \neq \phi$$