Lecture 10

- Carathéodory : conv X covered by X-simplices

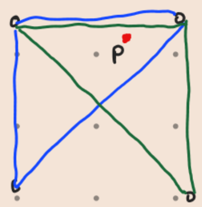
By Cara, we know that every pt in conv X is covered by some X-simplex. It is natural to ask if we always have a pt in conv X covered by many X-simplices.

Def: Given $X \subseteq \mathbb{R}^d$, the covering number of a pt p in X is

$m_X(p)$ = # d-simplices spanned by pts in X containing p.

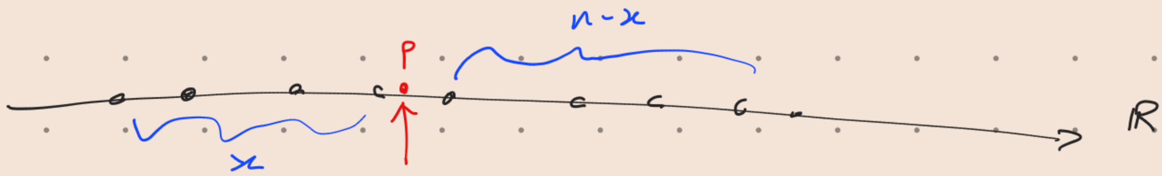
Q: Given any X, how large $m_X(p)$ we can always guarantee?

Ex $d=2$ $m_X(p)=2$



i.e. what is $\min_{|X|=n} \max_{p} m_X(p)$

Warm up : dim 1, n pts on $\mathbb{R}$



$$m_X(p) = x(n-x) \le \frac{n^2}{4} \approx \frac{1}{2}\binom{n}{2}$$

So is it true that for any dim. d, given any $X \subseteq \mathbb{R}^d$ we can always find a pt $p \in$ conv X covered by a positive fraction of X-simplices?

Planar case $d=2$

Boros & Füredi: proved the optimal fraction $2/9$ for $d=2$.

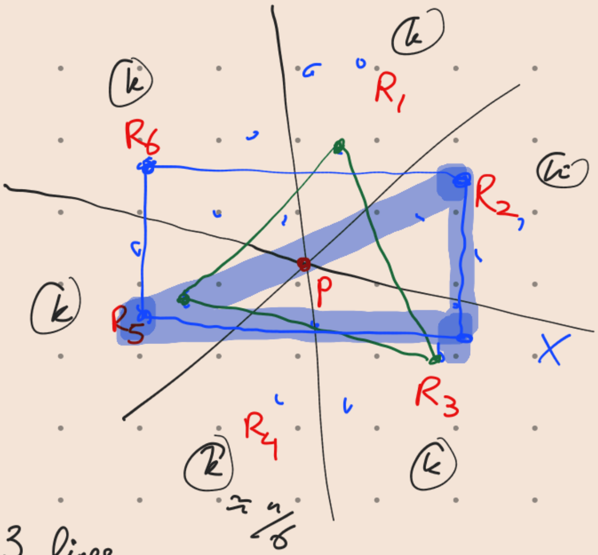
Thm 1. $\forall$ n-pt set $X \subseteq \mathbb{R}^2$ in general position, $\exists$ a pt covered by $\ge \frac{2}{9}\binom{n}{3} - O(n^2)$

triangles spanned by X.

Rmk The optimal fraction $c(d)$ is only known for $c(1) = 1/2$, $c(2) = 2/9$.

Open: $c(3) = ?$

Lem2 (Cedar) $\forall$ n-pts in general position on $\mathbb{R}^2$, there exist 3 concurrent lines s.t. all 6 regions have $\geq \frac{n}{6} - 1$ pts.



Pf (Thm1, due to Bukh)

- Let $k = n/6 - 1$, and take the 3 concurrent lines from Lem 2.

  Take p as the intersection pt of these 3 lines.

  Shall see that $\#\, X\text{-}\Delta s \ni p$ is $\geq 8k^3 = 8\left(\frac{n}{6}\right)^3 \approx \frac{2}{9}\binom{n}{3} - O(n^2)$.

(i) alternating Δs: $R_1 R_3 R_5$ $\Delta s \longrightarrow k^3$

$R_2 R_4 R_6 \longrightarrow k^3$

(ii) Skip antipodal pairs: Skip $R_1 R_4$, so $R_2 R_3 R_5 R_6$

exer. Claim $\forall$ interior pt of a convex 4-gon is contained by ≥ 2 Δs spanned by it vertices

$$\Rightarrow \geq \underset{\substack{\uparrow \\ \text{choices of antipodal} \\ \text{pairs to skip}}}{3} \cdot \overset{\Downarrow}{2k^4} \cdot \frac{1}{k} = 6k^3$$

$\frac{1}{k}$ $\leftarrow$ over counting, each such Δ is counted k times: as we can choose any of k pts from R_6 to form the conv 4-gon. (say $\in R_2 R_3 R_5$)

$\blacksquare$

We prove Lem 2 for continuous measure, a standard approx. argument yields the finite version above.

Lem (Cedar) $\forall$ nice measure μ on $\mathbb{R}^2$ w/ $\mu(\mathbb{R}^2) = 1$,

(nice: absolutely continuous w.r.t. the Lebesgue meas. w/ a positive continuous density.)

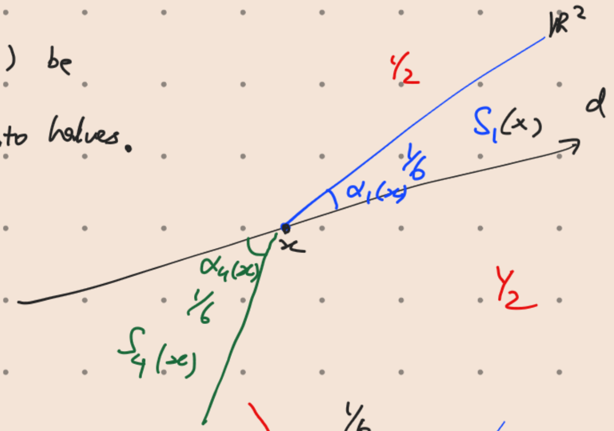
$\exists$ 3 concurrent lines s.t. all 6 regions have equal mass $\frac{1}{\ldots}$

3 concurrent lines ... all 6 regions have equal meas. $\frac{1}{6}$.

Pf: • For a direction $d \in S^1$, let $\ell(d)$ be the line in this direction cutting meas. into halves.

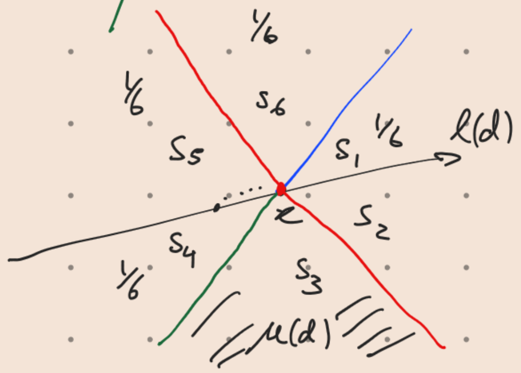
• For a $x \in \ell(d)$, let $S_1(x)$ be the unique region w/ meas. $1/6$.

and similar $S_4(x)$.



$$\alpha_1(x) \longrightarrow \begin{cases} 0 & \text{as } x \to -\infty \\ \pi & \text{as } x \to +\infty \end{cases}$$

$$\alpha_4(x) \longrightarrow \begin{cases} \pi & \text{as } x \to -\infty \\ 0 & \text{"} \ x \to +\infty \end{cases}$$



$\Rightarrow \exists$ a unique pt x s.t. $S_1(x)$ & $S_4(x)$ line up.

• Take a line cutting upper part into two equal meas $\Rightarrow \mu(S_5) = \mu(S_6) = \frac{1}{6}$

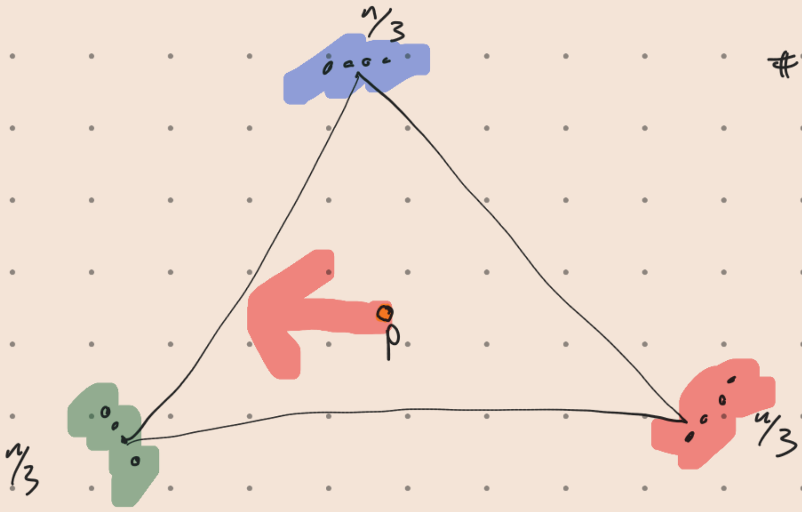
Let $\mu(d) = \mu(S_3)$ say $\mu(d) \neq 1/6$.

Note that $(\mu(d) - \frac{1}{6})(\mu(d+\pi) - \frac{1}{6}) < 0$

• Continuity of $\mu \Rightarrow \exists$ a direction d s.t. $\mu(d) = 1/6$. ∎

Tight construction for $d=2$ case

Ex 1.



rainbow Δ is $\left(\frac{n}{3}\right)^3 \approx \frac{2}{9}\binom{n}{3}$.

Turns out that $\exists p$ w/ $u_X(p) \geq \left(\frac{2}{9} + c\right)\binom{n}{3}$, $c > 0$.

Stretch grid example

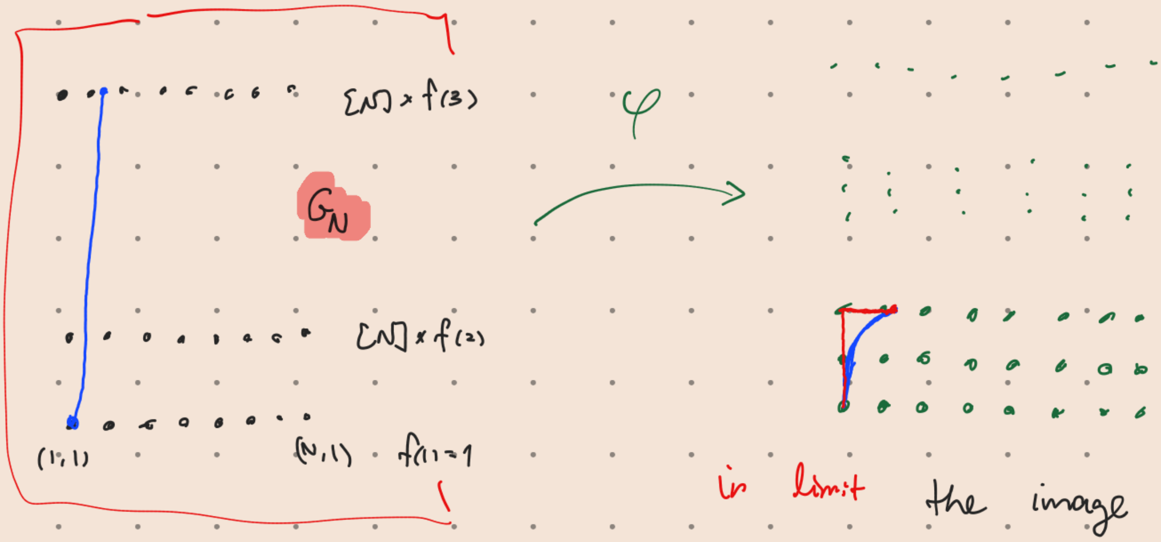
fast growing

Fix a funct. $f: [0,\infty) \to [0,\infty)$ (i.e. $f(1) \ll f(2) \ll f(3) \ll \dots$)

Consider the grid $G_N := \{1, \dots, N\} \times \{f(1), f(2), \dots, f(N)\}$ ($f(1) = 1$)

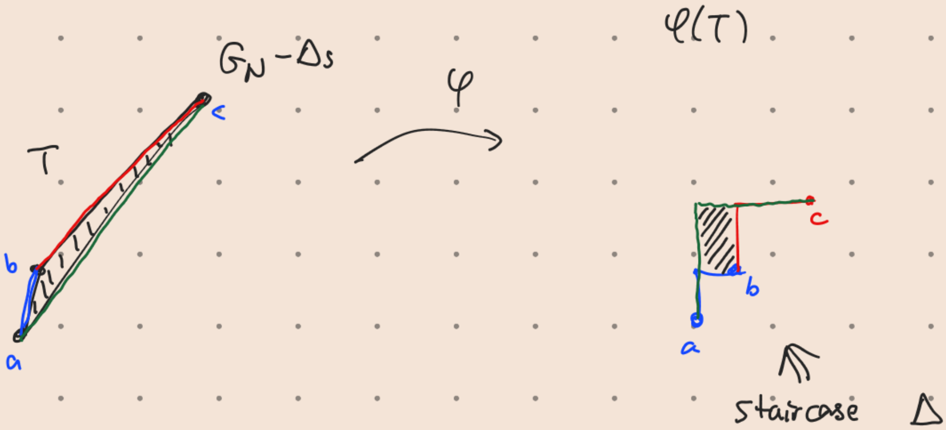
- Let φ be the map bring G_N to $[N] \times [N]$:

i.e. $\varphi : (x, y) \mapsto (x, f^{-1}(y))$



in limit the image of a line ℓ $\varphi(\ell)$ is a staircase path.

Δ_s becomes:



Tight example: Take diagonal in G_N: $(i, f(i))_{i \in [N]}$

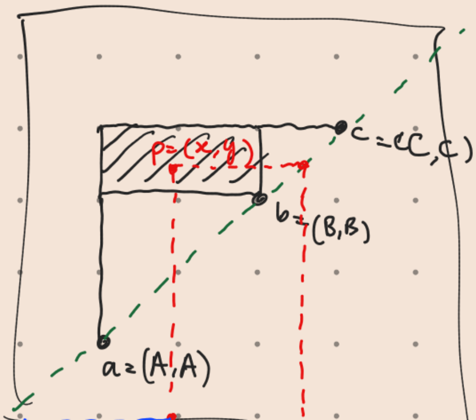
NTS $\forall p$, $\leq 2/9$ staircase Δs w./ endpts on diagonal containing p.

$\Delta abc \ni p$ only when

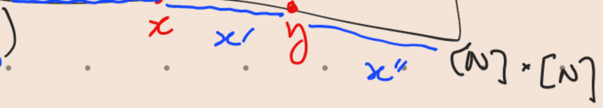
$a \in [0, x]$

$b \in [x, y]$

$c \in [y, N]$



$$\Longrightarrow \#\text{such } \Delta\text{s} \leq x \cdot \underbrace{(y-x)}_{x'} \cdot \underbrace{(N-y)}_{x''}$$



$x + x' + x'' = N$

$$\leq \left(\frac{N}{3}\right)^3 = \frac{2}{9}\binom{N}{3} + O(N^2).$$

∎