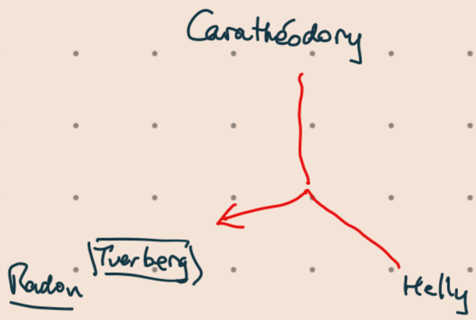
Lecture 8

Tverberg Thm Let $r \geq 2$, $X \subseteq \mathbb{R}^d$ a set of $(r-1)(d+1)+1$ pts.

$\Rightarrow \exists$ r-partition $X = X_1 \cup \dots \cup X_r$ s.t.

$$\bigcap_{i \in [r]} \operatorname{conv} X_i \neq \emptyset$$

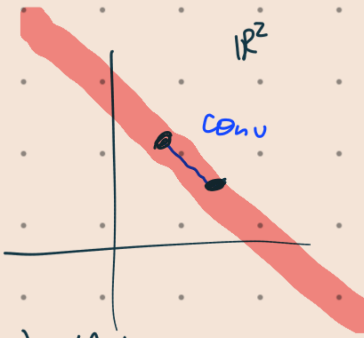


Rmk $r=2 \Rightarrow d+2$ pts $\Rightarrow$ Radon Lem.

Def. A set of pts $a_1, \dots, a_n \in \mathbb{R}^d$ are alg. indep. if all coord. of a_i (dn numbers in total) do not satisfy any polynomial w/ integer coeff.

Lem. If $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$, $n = (r-1)(d+1)$, is a set of alg. indep. pts, where $r \geq 2$, $d \geq 1$

$\Rightarrow \forall$ r-partition $A = A_1 \cup \dots \cup A_r$, $\bigcap_{i=[r]} \operatorname{aff} A_i = \emptyset$



Heuristic • As we consider affine hull in $\mathbb{R}^d$, may assume each $|A_i| \leq d+1$.

• $\dim(\operatorname{aff} A_i) = |A_i| - 1 \Rightarrow \operatorname{codim}(\operatorname{aff} A_i) = (d+1) - |A_i|$

So $\operatorname{aff} A_i$ = intersection of $(d+1) - |A_i|$ hyperplanes.

• $\bigcap_{i \in [r]} \operatorname{aff} A_i$ = intersection of $\sum_{i \in [r]} \left[(d+1) - |A_i|\right]$ hyperplanes

$$= r(d+1) - n = d+1$$

But $\bigcap$ of $d+1$ hyperplanes is empty due to alg. indep.

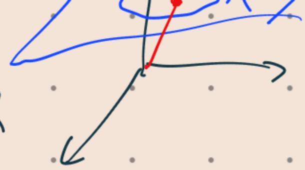
Tverberg cone version • $X \subseteq \mathbb{R}^{d+1}$
• $|X| = (r-1)(d+1)+1$
• $0 \notin \operatorname{conv} X$

$\Rightarrow \exists\ X = X_1 \cup \dots \cup X_r$ s.t. $\bigcap_{i \in [r]} \operatorname{cone} X_i \neq \{0\}$

• Cone $\Rightarrow$ original: $\mathbb{R}^d \to \mathbb{R}^{d+1}$ as $x_{d+1} = 1$

• $0 \notin \operatorname{conv} X$

cone $\Rightarrow$ $X = X_1 \cup \cdots \cup X_r$ s.t.



every cone X_i

$\bigcap_{i \in [r]} \text{cone } X_i$ contains a ray R

Not hard to see that $R \cap \{x_{d+1} = 1\}$ is a Tverberg pt.

Idea: Tensor prod. trick to lift pts to higher dim & apply colorful Carathéodory there.

[Sarkaria]

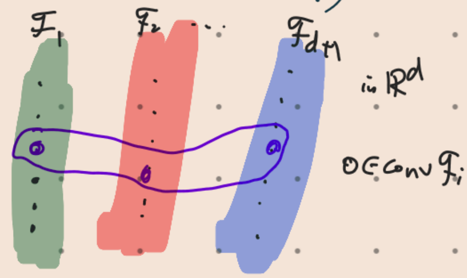
Pf: Say $X = \{x_1, \dots, x_{n+1}\}$ where $n = (r-1)(d+1)$

Tverberg cone version
- $X \in \mathbb{R}^{d+1}$
- $|X| = (r-1)(d+1)+1$
- $0 \notin \text{conv } X$

$\Rightarrow$ $\exists$ $X = X_1 \cup \cdots \cup X_r$ s.t. $\bigcap_{i \in [r]} \text{cone } X_i \neq \{0\}$

- Lift each pt of X from $\mathbb{R}^{d+1}$ to $\mathbb{R}^n = \mathbb{R}^{(r-1)(d+1)}$ via tensor.
  - Take $v_1, \dots, v_r \in \mathbb{R}^{r-1}$, where $v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, v_{r-1} = e_{r-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

    and $v_r = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} = -\sum_{i=1}^{r-1} e_i$



  - Lift $x_i \longrightarrow A_i \subseteq \mathbb{R}^n$ where

$$A_i = \{ v_1 \otimes x_i, \dots, v_r \otimes x_i \}$$

$$= \left\{ \begin{pmatrix} x_i^T \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ x_i^T \end{pmatrix}, \begin{pmatrix} -x_i^T \\ \vdots \\ -x_i^T \end{pmatrix} \right\}$$

Note that $\sum_{a \in A_i} a = 0 \Rightarrow 0 \in \text{conv } A_i$, $i = 1, \dots, n+1$

$A_i \subseteq \mathbb{R}^n$

$\xRightarrow{\text{colorful Carathéodory}}$ $\exists$ transversal $a_1 \in A_1, \dots, a_{n+1} \in A_{n+1}$ s.t.

$0 \in \text{conv}\{a_1, \dots, a_{n+1}\}$

(*) $\cdots = \text{conv}\{ v_{f(1)} \otimes x_1, v_{f(2)} \otimes x_2, \dots, v_{f(n+1)} \otimes x_{n+1} \}$

- $f : [n+1] \longrightarrow [r]$ yields an r partition of $[n+1]$

$$[n+1] = f^{-1}(1) \cup \cdots \cup f^{-1}(r)$$

$$\Updownarrow$$

$$X = X_1 \cup \cdots \cup X_r$$

i.e. $X_k = \{ x_i : f(i) = k \}$.

- (*) $\Rightarrow$ $\exists$ $\alpha_i \geq 0$ ... $\sum^{n+1}$ ... $\sum^{n+1}$

$\sum_{i=1}^{n+1} \alpha_i = 1$. s.t. $0 = \sum_{i=1} \alpha_i a_i = \sum_{i=1} \alpha_i \, v_{f(i)} \otimes x_i$

$$= \sum_{k=1}^{r} \sum_{i \in f^{-1}(k)} \alpha_i \, v_k \otimes x_i$$

i.e. $f(i) = k$

$$0 = \sum_{k=1}^{r} v_k \otimes \boxed{\sum_{i: f(i)=k} \alpha_i x_i} =: u_k$$

Note $u_k = \sum_{i: f(i)=k} \alpha_i x_i \in \text{cone } X_k$

$$0 = \sum_{k=1}^{r} v_k \otimes u_k = \begin{pmatrix} -u_1^T- \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_2^T- \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ -u_{r-1}^T- \end{pmatrix} + \begin{pmatrix} -u_r^T \\ -u_r^T \\ \vdots \\ -u_r^T \end{pmatrix}$$

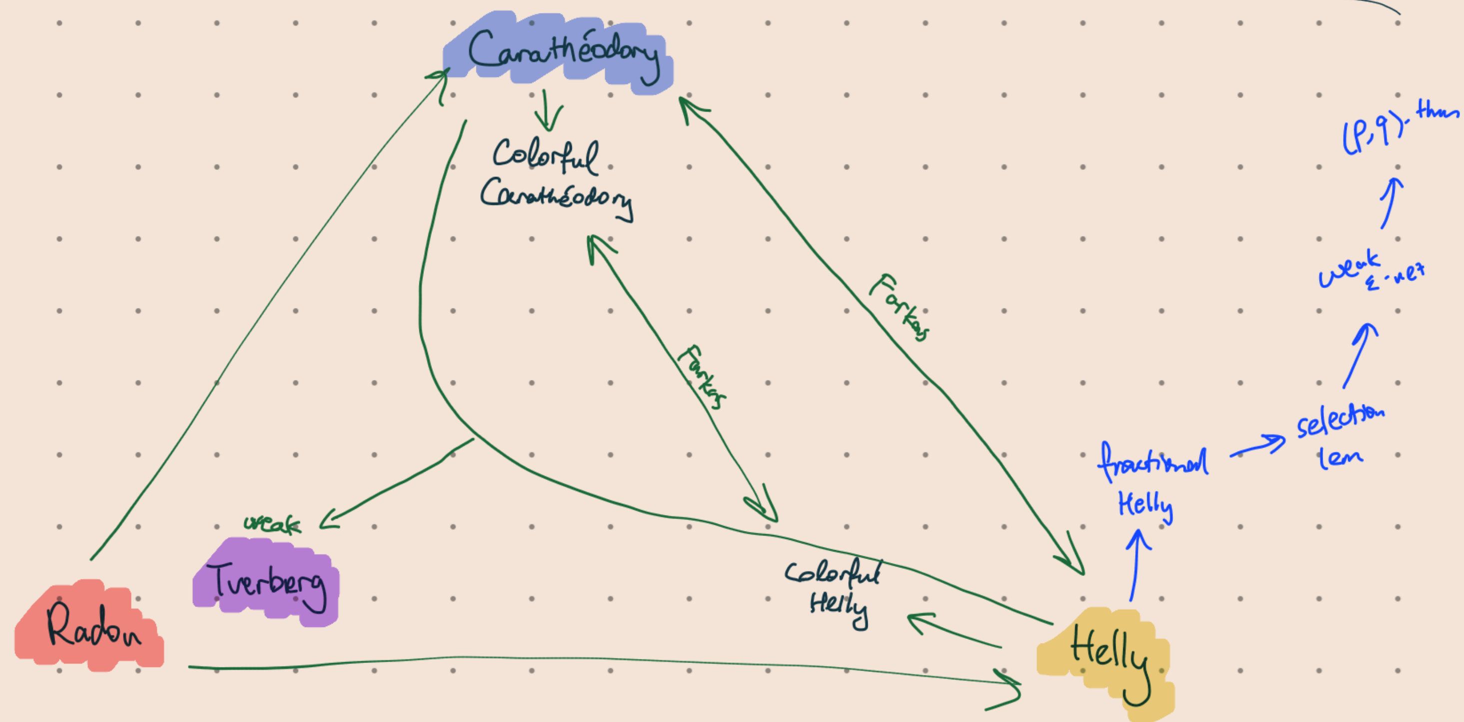
$\Rightarrow u_1 = u_2 = \cdots = u_r =: z$

i.e. $z \in \bigcap_{k \in [r]} \text{cone } X_k$

- Left to show $z \neq 0$.

Recall $0 \notin \text{conv } X$. If $z = 0$, $0 \in \text{cone } X_1$

$\Rightarrow 0 \in \text{conv } X_1 \subseteq \text{conv } X$ ↯ ∎



$\forall d \geq 1, \alpha > 0, \exists \beta = \beta(\alpha, d)$ s.t. TFH.

Fractional Helly. Finite family $\mathcal{F}$ of conv sets in $\mathbb{R}^d$,

if ~~every~~ α-fraction of (d+1)-tuple $T \in \binom{\mathcal{F}}{d+1}$ ~~is~~ are intersecting

β-fraction of $\mathcal{F}$ i.e. $\exists\, \mathcal{F}' \subseteq \mathcal{F}$

then $\mathcal{F}$ is intersecting, i.e. $\bigcap \mathcal{F} \neq \phi$. $|\mathcal{F}'| \geq \beta |\mathcal{F}|$

$\bigcap \mathcal{F}' \neq \phi$.

Rmk Best possible $\beta = 1 - (1-\alpha)^{\frac{1}{d+1}}$. $\beta \to 1$ as $\alpha \to 1$.