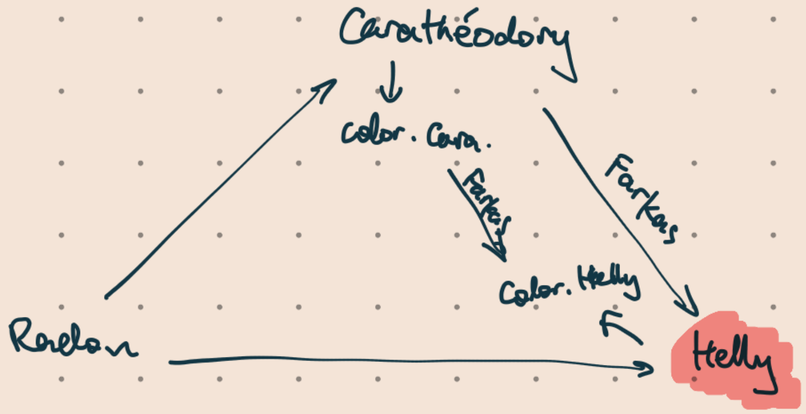
Lecture 6

Recall



§ ∞-version Helly

Thm. • $\mathcal{F}$ a collection of compact convex sets in $\mathbb{R}^d$

if all $(d+1)$-tuples in $\mathcal{F}$ are intersecting $\Rightarrow$ $\mathcal{F}$ is intersecting.

Rmk: • ∞-ver. is not true if sets are not compact.

Ex $\mathbb{R}^1$: not closed: $\{(0, 1/n)\}_{n\in\mathbb{N}}$

not bounded: $\{[n, \infty)\}_{n\in\mathbb{N}}$

• Slightly weaker hypothesis suffices: all sets closed

1 set compact.

Pf • For any fixed n: Consider $\{C_1, \dots, C_n\}$ finite fam.

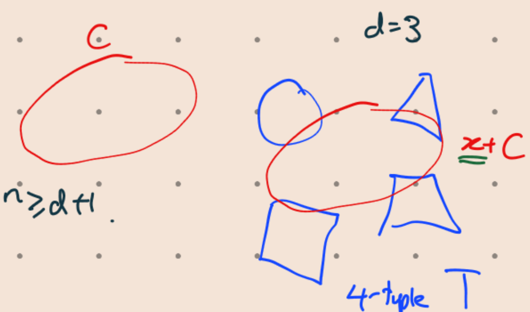
(finite) Helly $\Longrightarrow$ $\exists\, x_n \in \bigcap_{i\in[n]} C_i$

• All pts $x_1, x_2, \dots, x_n, \dots \in C_1$ compact

$\Rightarrow$ converging subseq $\xrightarrow{\text{limit}} x \in C_1$.

Finally $x \in C_i$ $\forall i$ as C_i is closed. ∎

§ Applications of Helly



Thm • $\mathcal{F}$ finite fam. of convex sets in $\mathbb{R}^d$, $|\mathcal{F}| = n \geq d+1$.

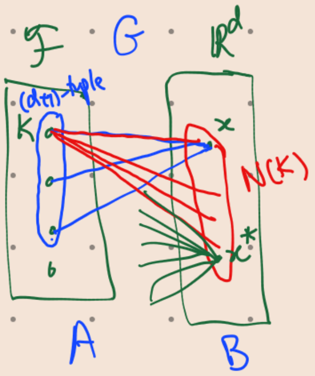
• C conv. set in $\mathbb{R}^d$

If $\forall$ $(d+1)$-tuple $T \in \binom{\mathcal{F}}{d+1}$, $\exists$ a translate of C intersecting all members of T $\Longrightarrow$ $\exists$ a translate of C intersecting every member of $\mathcal{F}$.

Thinking

- try identify translate $x+C$ by pts $x \in \mathbb{R}^d$

bipartite aux. graph



an edge btw K & x if $K \cap (x+C) \neq \phi$

Hyp $\Rightarrow$ $\forall$ $(d+1)$-set $S \subseteq A$, $N_G(S) \neq \phi$

$N(K) = \{x \in \mathbb{R}^d : (x+C) \cap K \neq \phi\} = K^*$

$\{K^* : K \in \mathcal{F}\} = \mathcal{F}^*$

$\forall$ $(d+1)$-tuple of $\mathcal{F}^*$ intersecting.

Pf.: • $\forall K \in \mathcal{F}$, def: $K^* = \{x \in \mathbb{R}^d : (x+C) \cap K \neq \phi\}$

then [$\forall$ $(d+1)$-tuple T of K, $\exists$ $x+C$ intersecting T]

$\Rightarrow$ $\forall$ $(d+1)$-tuple of $\mathcal{F}^* := \{K^* : K \in \mathcal{F}\}$ is intersecting

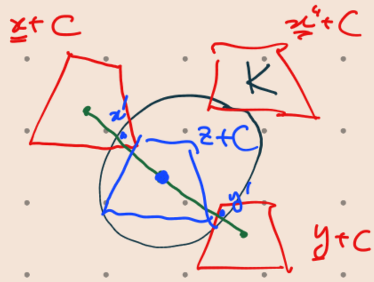
- If Helly's thm applies $\Rightarrow$ $\exists$ $x^* \in \bigcap \mathcal{F}^*$

This is precisely saying $x^* + C$ intersects all members of $\mathcal{F}$.

- Left to show that all K^* is convex. (So Helly applies)

Pf: $\forall x, y \in K^*$, $\forall \alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$

NTS: $\alpha x + \beta y \in K^*$



- $x, y \in K^* \Rightarrow$ $\exists$ $x' \in (x+C) \cap K \neq \phi$

$\exists$ $y' \in (y+C) \cap K \neq \phi$

- Convexity of K, $x', y' \in K$ $\Rightarrow$ $\alpha x' + \beta y' \in K$

$x' \in x+C$
$y' \in y+C$
- $\Rightarrow$ $\alpha x' + \beta y' \in \alpha x + \alpha C + \beta y + \beta C = \alpha x + \beta y + C$

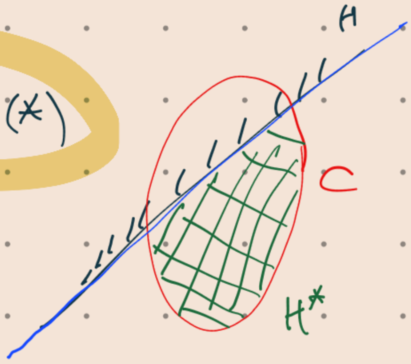
$\alpha x' + \beta y' \in (\alpha x + \beta y + C) \cap K$

In particular $\alpha x + \beta y \in K^*$. ∎

Thm • $\mathcal{H}$ finite fam open (closed) half-spaces in $\mathbb{R}^d$ w/ $|\mathcal{H}| \geq d+1$.

• C convex sets in $\mathbb{R}^d$.

If $C \subseteq \cup\mathcal{H} \Rightarrow \exists\, \mathcal{H}' \subseteq \mathcal{H}$ $\begin{cases} |\mathcal{H}'| = d+1 \\ C \subseteq \cup\mathcal{H}' \end{cases}$.



Pf: • $C \subseteq \cup\mathcal{H} \Rightarrow C \cap \Big(\underbrace{\bigcap_{H\in\mathcal{H}} H^c}_{\overline{\cup\mathcal{H}}}\Big) = \phi \cdots (*)$

• $\forall H \in \mathcal{H}$, define $H^* = C \cap \underbrace{H^c}_{\text{convex}}$ $= C \setminus H$

Let $\mathcal{F} = \{H^* : H \in \mathcal{H}\}$

• $(*) \Rightarrow \bigcap_{H\in\mathcal{H}} H^* = \phi = \bigcap\mathcal{F}$. We need to show H^* is convex.

H half-space $\Rightarrow H^c$ convex $\Rightarrow C \cap H^c = H^*$ convex.

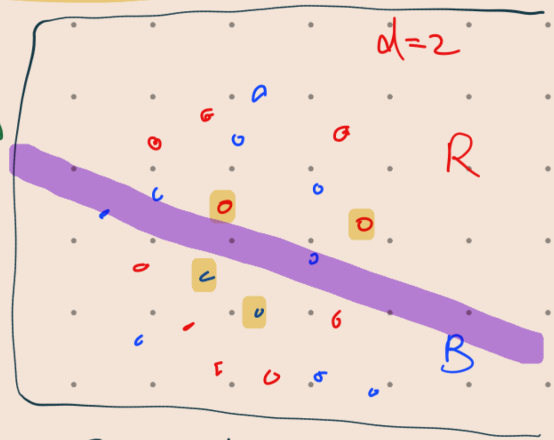
then Helly (contrapositive) applies $\Rightarrow \exists\, |\mathcal{H}'| = d+1$, $\mathcal{H}' \subseteq \mathcal{H}$ s.t. $\bigcap_{H\in\mathcal{H}'} H^* = \phi$.

$\Leftrightarrow C \cap \Big(\bigcap_{H\in\mathcal{H}'} H^c\Big) = \phi \Leftrightarrow C \subseteq \cup\mathcal{H}'$. ∎

Thm [Kirchberger 1903]

• R finite set of red pts in $\mathbb{R}^d$

• B —"— blue —"—



if $\forall\, Y \subseteq R \cup B$ of size $|Y| \leq d+2$ $\Rightarrow$ $\exists$ a hyperplane strictly separating R and B.

$\exists$ a hyperplane strictly separating $Y \cap R$ & $Y \cap B$

Pf: • We want a hyperplane $a \cdot x = b$ ~~s.t.~~ strictly separates R & B.

i.e. $\forall\, y \in R$, $\forall\, z \in B$ $\Rightarrow a\cdot y < b < a\cdot z$

$\Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} y \\ -1 \end{pmatrix} < 0 < \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}$

• $\forall\, y \in R$, define half-space

$C_y = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{d+1} : \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} y \\ -1 \end{pmatrix} < 0 \right\}$

$\forall\, z \in B$: $C_z = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{d+1} : \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} z \\ -1 \end{pmatrix} > 0 \right\}$

Let $\mathcal{F} = \{ C_x : x \in R \cup B \}$ finite fam of convex sets in $\mathbb{R}^{d+1}$.

- Hyp. $\Rightarrow$ $\forall$ $(d+2)$-tuple of $\mathcal{F}$ is intersecting.
  - Helly $\Rightarrow$ $\cap \mathcal{F} \neq \emptyset$

    i.e. $\exists$ a pt $\begin{pmatrix} a^* \\ b^* \end{pmatrix} \in \mathbb{R}^{d+1}$ piercing all half-spaces $C_x \in \mathcal{F}$.

$\Leftrightarrow$ $a^* \cdot x = b^*$ strictly seperates R & B. $\blacksquare$

Next time :
- Center pt thm.
- Jung's thm.