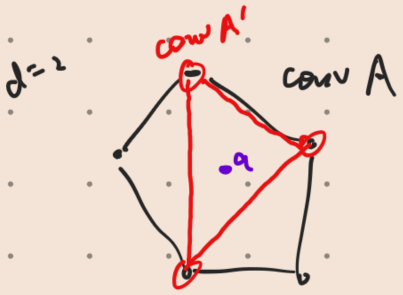
Lecture 5

常帆 : Carathéodory : $\forall A \subseteq \mathbb{R}^d$, $\forall a \in \operatorname{conv} A$

$\forall \varepsilon > 0$



$\Longrightarrow \exists A' \subseteq A$ s.t. $\begin{cases} |A'| \le d+1 \\ a \in \operatorname{conv} A' \end{cases}$

$C(\varepsilon)$

$\operatorname{dist}(a, \operatorname{conv} A') \le \varepsilon$

- $d+1$ cannot be improved.
- Is there a dim-free version?

§ Colorful Helly

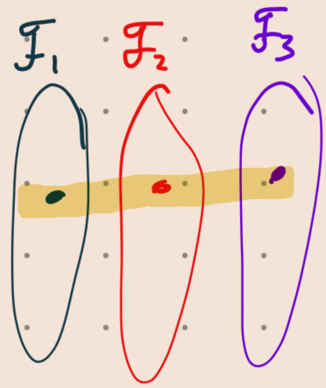
Thm (?) Given finite fam of conv sets $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$ in $\mathbb{R}^d$,

if $\forall$ transversal / rainbow $(d+1)$-tuple

$\forall C_i \in \mathcal{F}_i$, $i \in [d+1]$

is intersecting: $\bigcap_{i \in [d+1]} C_i \neq \emptyset$

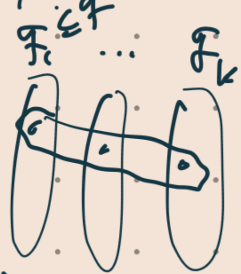
$\Longrightarrow$ $\exists i \in [d+1]$ s.t. $\mathcal{F}_i$ is intersecting: $\bigcap \mathcal{F}_i \neq \emptyset$



Rmk:
- If we take $\mathcal{F}_1 = \dots = \mathcal{F}_{d+1} = \mathcal{F} \Longrightarrow$ Helly
- Helly number of $\mathcal{F}$ = min h : if $\forall$ h-tuple of $\mathcal{F}$ inter. $\Rightarrow \bigcap \mathcal{F} \neq \emptyset$

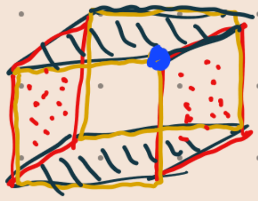
$\boxed{\text{Helly Thm}}$: Helly # of conv sets in $\mathbb{R}^d = d+1$

- Colorful Helly # of $\mathcal{F}$ = min k : $\forall$ finite $\mathcal{F}_1, \dots, \mathcal{F}_k \subseteq \mathcal{F}$ if all transversal intersecting $\Rightarrow$ One color $\mathcal{F}_i$ is intersecting.



$\boxed{\text{Colorful Helly}}$: Colorful Helly # of conv sets in $\mathbb{R}^d = d+1$

Example [colorfully Helly # >> Helly #]



Let $\mathcal{F}$ be the fam. of axis-aligned boxes in $\mathbb{R}^3$

$\mathcal{F}_1$ $\mathcal{F}_2$ $\mathcal{F}_3$

$\boxed{\text{Ex. Helly \# } = 2}$

Colorful Helly # > 3

In general : axis-aligned boxes in $\mathbb{R}^d$ $\begin{cases} \text{Helly \#} = 2 \\ \text{colorful Helly \#} = d+1 \end{cases}$

Pf [Colorful Helly]

- May assume all conv sets are polytope.
- Take a colorful d-tuple

$$\tilde{\mathcal{H}} = \{ A_1, \dots, A_d \}$$

s.t. $$a \cdot p_{\tilde{\mathcal{H}}} = \min_{\substack{\text{colorful} \\ d\text{-tuple } \mathcal{H}}} a \cdot p_{\mathcal{H}}$$

W.l.o.g. may assume $A_i \in \mathcal{F}_i$.

Claim: $p_{\tilde{\mathcal{H}}} \in \bigcap \mathcal{F}_{d+1}$

pf Apply Lem to $\{A_1, \dots, A_d, C\} = \mathcal{H}$ $\forall C \in \mathcal{F}_{d+1}$

($\tilde{\mathcal{H}}$ is the $\mathcal{H}^*$) $\overset{(i)}{\Rightarrow}$ $p_{\tilde{\mathcal{H}}} = p_{\mathcal{H}} \in \bigcap \mathcal{H}$

$\Rightarrow$ $p_{\tilde{\mathcal{H}}} \in C$ $\forall C \in \mathcal{F}_{d+1}$. ■

$\forall \mathcal{H} \in \mathcal{I}(\mathcal{F})$ direction

$\exists$ unique max $p_{\mathcal{H}}$: $\displaystyle\max_{x \in \mathcal{H}} a \cdot x$

$M = \{ p_{\mathcal{H}} : \mathcal{H} \in \mathcal{I}(\mathcal{F}) \}$

$\exists$ uniq. min $p_{\mathcal{H}^*}$ ("1st max")

Lem (i) $\forall \mathcal{H} \supseteq \mathcal{H}^*$, $\mathcal{H} \in \mathcal{I}(\mathcal{F})$

$\Rightarrow$ $p_{\mathcal{H}} = p_{\mathcal{H}^*}$

(ii) assuming Helly for $\mathbb{R}^{d-1}$

$\Rightarrow$ $|\mathcal{H}^*| \leq d$

§ 3rd pf of (colorful) Helly via Carathéodory

& Farkas.

Thm [Multiple Separation]

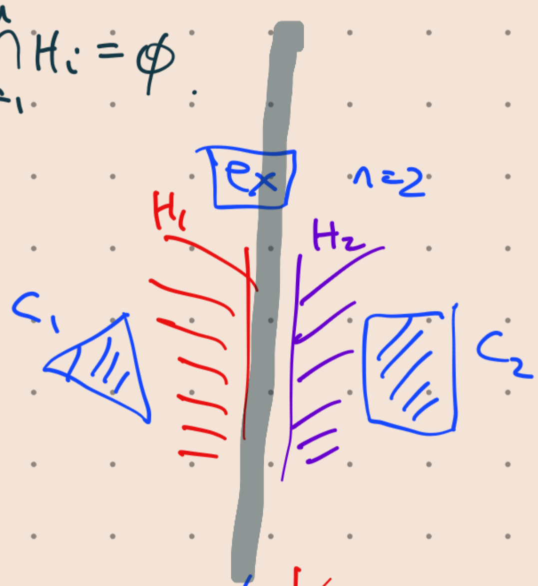
Let $C_1, \dots, C_n$ be closed convex sets in $\mathbb{R}^d$, $n \geq 2$

and C_1 is compact

Then $\bigcap_{i \in [n]} C_i = \phi \iff$ $\exists$ closed half-spaces $H_1, \dots, H_n$ s.t.

- $C_i \subset H_i$
- $\bigcap_{i=1}^{n} H_i = \phi$.

Pf: ($\Leftarrow$) trivial

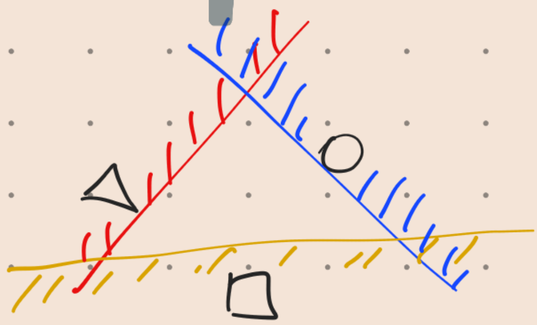


($\Rightarrow$) $n=2$ is separation thm.

Induction on n.

Illustrate w/ $n=3$. (general n exer.)

- $\underbrace{(C_1 \cap C_2)}_{\text{compact}} \cap \underbrace{(C_3)}_{\text{closed}} = \phi$



by $n=2$ case $\Rightarrow \exists$ closed half-space $H_3 \supseteq C_3$

s.t. $(C_1 \cap C_2) \cap H_3 = \phi$

- $\underbrace{(C_1 \cap H_3)}_{\text{Compact}} \cap \underbrace{C_2}_{\text{closed}} = \phi$

$\Rightarrow \exists H_2 \supseteq C_2$ s.t. $(C_1 \cap H_3) \cap H_2 = \phi$

- $\underbrace{C_1}_{\text{compact.}} \cap \underbrace{(H_2 \cap H_3)}_{\text{closed}} = \phi \Rightarrow \exists H_1 \supseteq C_1$ closed half-space s.t. $\bigcap_{i \in [3]} C_i = \phi$. ∎

Pf [(Colorful) Helly] Contrapositive $\bigcap \mathcal{F} = \phi$

$\bigcap_{i \in [n]} C_i = \phi \Rightarrow \exists I \in \binom{[n]}{d+1}$ s.t. $\bigcap_{i \in I} C_i = \phi$

B ... and ... all conv sets

Helly) • By polytope reduction trick ⇒ we ... are polytope compact

$\cap \mathcal{F}_1 = \emptyset \dots \cap \mathcal{F}_{d+1} = \emptyset \Rightarrow \exists$ transversal $C_i \in \mathcal{F}_i$, $\bigcap_{i\in[d+1]} C_i = \emptyset$

- So by multiple separation thm, we have closed half-spaces

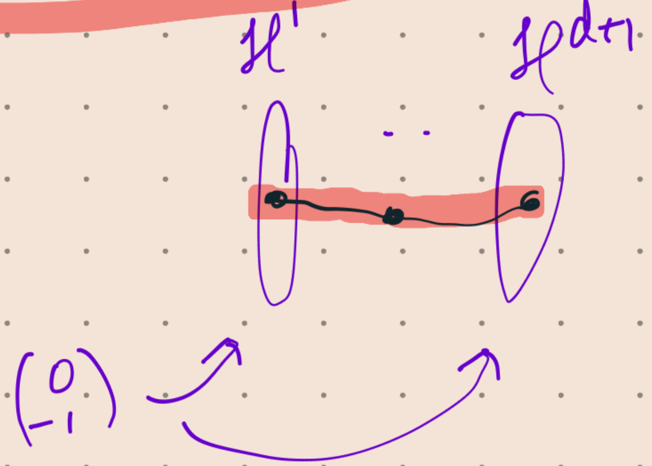
$$H_i \supseteq C_i \quad \forall i \in [n] \quad \text{s.t.} \quad \boxed{\bigcap_{i\in[n]} H_i = \emptyset}.$$

$$H_i := \{x \in \mathbb{R}^d : a_i \cdot x \le b_i\} \text{ for some } a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R}.$$

$\cap \mathcal{F}_1 = \emptyset$

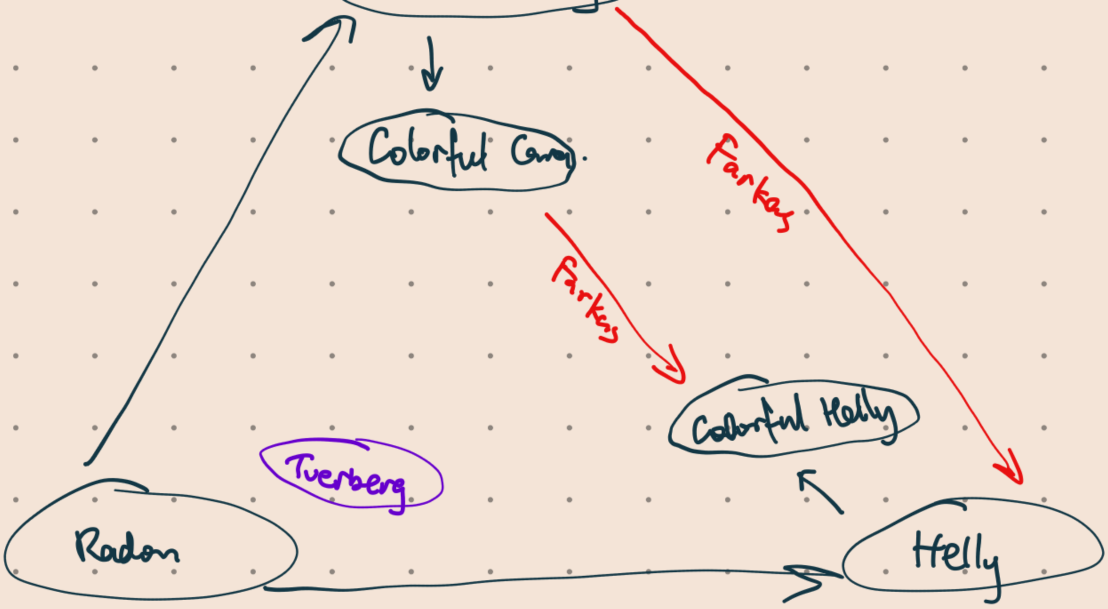
$$\begin{tikzcd}
\bigcap_{i\in[n]} H_i^{!} = \emptyset \arrow[r, Leftrightarrow, "\text{Farkas}"] \arrow[d, "?\ ☺\ ?"'] & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \in \operatorname{cone}\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \mathbb{R}^{d+1},\ i \in [n] \right\} \arrow[d, Rightarrow, "\text{Colorful Carathéodory (cone)}"] \\
\exists I \in \binom{[n]}{d+1} \text{ s.t. } \bigcap_{i\in I} H_i = \emptyset \arrow[r, Leftrightarrow, "\text{Farkas}"] & \exists I \in \binom{[n]}{d+1} \text{ s.t. } \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \operatorname{Cone}\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix} : i \in I \right\}
\end{tikzcd}$$

$$\bigcap_{i\in I} C_i = \emptyset \Leftarrow \bigcap_{i\in I} H_i = \emptyset$$



Exer: Prove Colorful Helly using this method.

Carathéodory


Colorful
Farkas
Farkas
Colorful Helly
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Radon
Helly