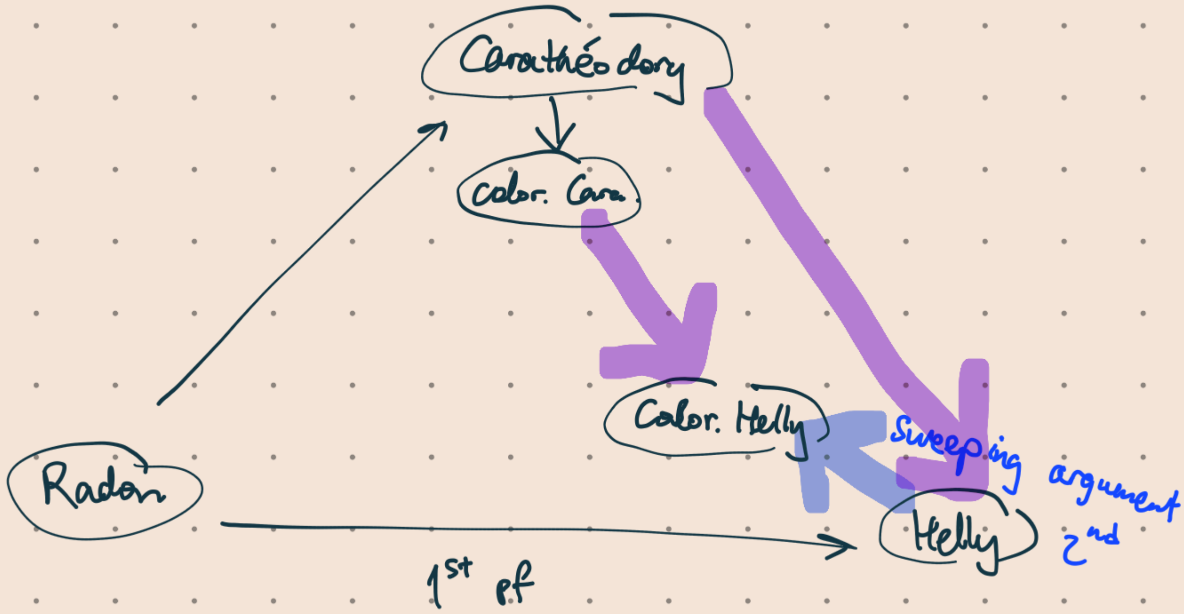
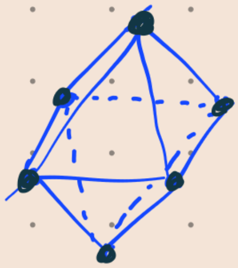
# Lecture 4

**Recap**




**Def**: Polytope: conv hull of a finite set of pts



## § Reduction to polytopes trick

For statements involving intersection patterns of convex sets such as Helly's thm, may assume we are dealing w/ compact sets, or even polytope using this trick.

**Example** Helly's thm hypothesis: every $(d+1)$-tuple intersects
$C_1, \dots, C_n$ conv. sets in $\mathbb{R}^d$

$$\forall I \in \binom{[n]}{d+1}, \quad \bigcap_{i \in I} C_i \neq \phi$$

Choose a pt $x_I \in \bigcap_{i \in I} C_i$

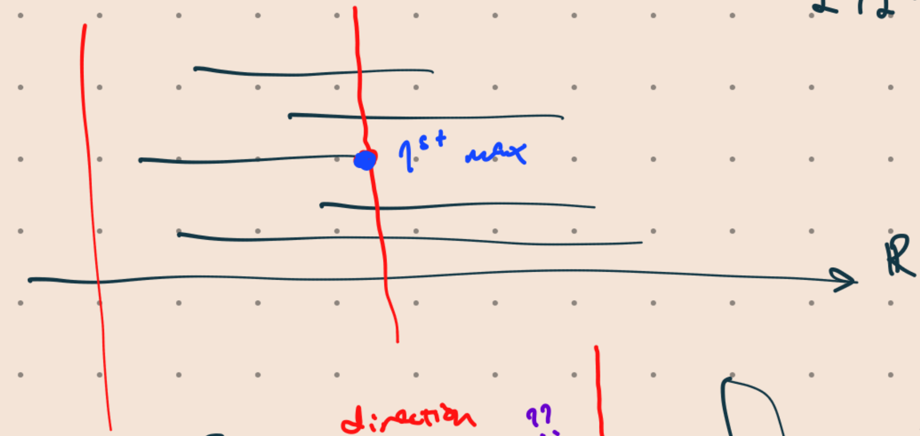
New sets (polytopes): define $K_i \subseteq C_i$

$$K_i = \operatorname{conv}\{x_I : i \in I\} \quad \text{polytope}$$

$$\bigcap_{i \in I} K_i \neq \phi \quad \text{as} \quad x_I \in \bigcap_{i \in I} K_i.$$

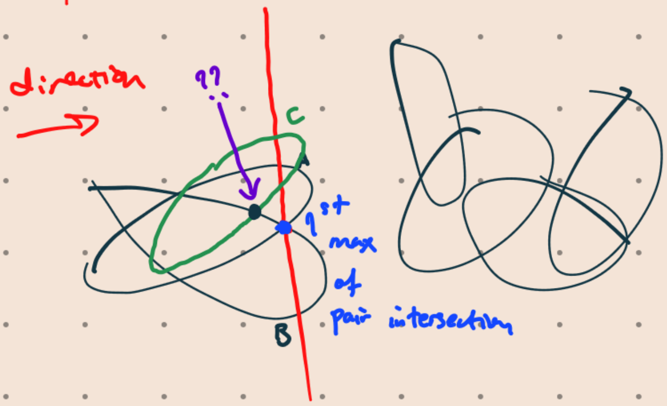
Recall Helly's thm in 1-dim: intervals $I_1, \dots, I_n$

$I \neq I'$, $I \cap I' \neq \phi \Rightarrow \bigcap_{i \in [n]} I_i \neq \phi$




Idea: find "1st max"

Try $\mathbb{R}^2$




§ Sweeping argument (2nd pf of Helly)

Notation • $\mathcal{F} = \{C_1, \dots, C_n\}$ conv. sets in $\mathbb{R}^d$.

• $\mathcal{I}(\mathcal{F})$ = all intersecting subfam. of $\mathcal{F}$

i.e. $= \{\mathcal{H} \in \mathcal{F} : \bigcap \mathcal{H} \neq \phi\}$ $\quad \bigcap \mathcal{H} = \bigcap_{C \in \mathcal{H}} C$

• By polytope reduction trick, may assume all $C_i$ are polytopes.

• As there are finitely many vertices in total,

we may choose a direction $a \in \mathbb{R}^d$ s.t.

$\forall \mathcal{H} \in \mathcal{I}(\mathcal{F})$ has a unique max in direction $a$.

i.e. $\boxed{\max_{x \in \bigcap \mathcal{H}} a \cdot x}$ is uniquely achieved.

Call this unique max pt $\boxed{p_{\mathcal{H}}}$

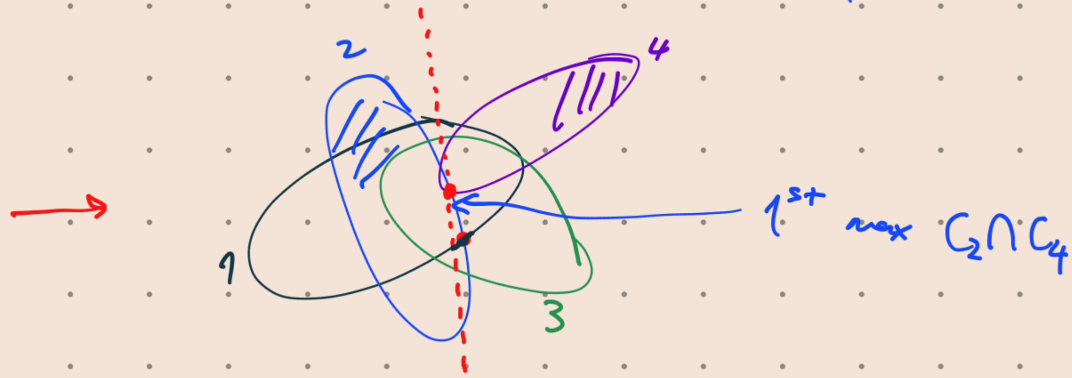
• Write $M = \{ p_{\mathcal{H}} : \mathcal{H} \in \mathcal{I}(\mathcal{F}) \}$

for the set of all max pts

Again (by rotating if necessary), we may assume

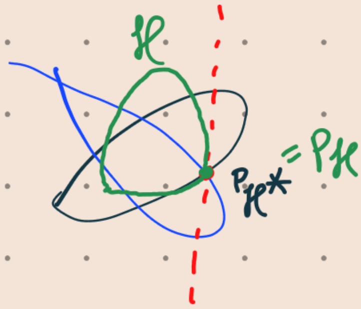
there is a unique $p_{\mathcal{H}^*} \in M$ minimising $a \cdot x$ over $x \in M$.

$1^{st}$ max




<u>Lem</u> (i) $\forall \mathcal{H} \in \mathcal{I}(\mathcal{F})$, if $\mathcal{H} \supseteq \mathcal{H}^* \implies p_{\mathcal{H}} = p_{\mathcal{H}^*}$

(ii) $|\mathcal{H}^*| \le d$ assuming Helly's thm in $\mathbb{R}^{d-1}$.




$\mathcal{H}^*$ = black & blue

$\mathcal{H}$ = $\mathcal{H}^*$ & green

<u>Pf</u>: (i)

• $a \cdot p_{\mathcal{H}} = \max_{x \in \cap \mathcal{H}} a \cdot x \le \max_{x \in \cap \mathcal{H}^*} a \cdot x = a \cdot p_{\mathcal{H}^*}$

As $\mathcal{H} \supseteq \mathcal{H}^*$, $\cap \mathcal{H} \subseteq \cap \mathcal{H}^*$

larger domain

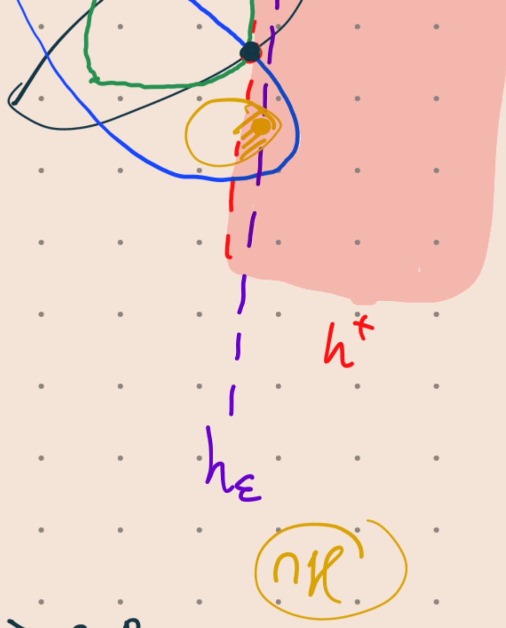
• minimality of $p_{\mathcal{H}^*} \Rightarrow$ "$=$" holds above!

As min in $M$ ($p_{\mathcal{H}^*}$) is unique $\Rightarrow$ (i)

(ii) Suppose not, say $\mathcal{H}^* = \{A_1, \dots, A_{d+1}\}$

• Let $h := \{x \in \mathbb{R}^d : a \cdot x = a \cdot p_{\mathcal{H}^*}\}$

$h^+ := \{x \in \mathbb{R}^d : a \cdot x \ge a \cdot p_{\mathcal{H}}\}$




<u>Obs</u>: $\cap\mathcal{H}^*$ is disjoint from interior of $h^+$ as $P_{\mathcal{H}^*}$ is the max pt.

1 dim lower

$\rightarrow h_\varepsilon := \{x \in \mathbb{R}^d : a\cdot x = a\cdot P_{\mathcal{H}^*} + \varepsilon\}$
$\subseteq \mathbb{R}^{d-1}$

- $\forall$ d-tuple $\mathcal{H} \subsetneq \mathcal{H}^* \Longrightarrow a\cdot P_{\mathcal{H}} > a\cdot P_{\mathcal{H}^*}$

  minimality of $P_{\mathcal{H}^*}$

(♡) So $\boxed{(\cap\mathcal{H}) \cap h_\varepsilon \neq \phi}$ (by choosing $\varepsilon$ small enough which is possible as there are finitely many members in $\mathcal{I}(\mathcal{F})$)

- Now apply Helly $(\mathbb{R}^{d-1})$ on $\{A \cap h_\varepsilon : A \in \mathcal{H}^*\}$

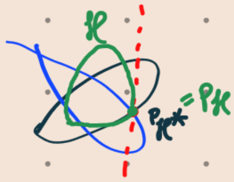
(♡) $\Rightarrow$ all d-tuple intersects $\Longrightarrow (\cap\mathcal{H}^*) \cap h_\varepsilon \neq \phi$

contradicting to the observation above. ▨

2nd
<u>Pf of Helly</u>

**Lem** (i) $\forall \mathcal{H} \in \mathcal{I}(\mathcal{F})$, if $\mathcal{H} \supseteq \mathcal{H}^* \Longrightarrow P_{\mathcal{H}} = P_{\mathcal{H}^*}$

(ii) $|\mathcal{H}^*| \le d$ assuming Helly's thm in $\mathbb{R}^{d-1}$.




- Induction on dim d.
  $d=1$ ☺
- Inductive step:
  - Lem (ii) $\Rightarrow |\mathcal{H}^*| \le d$

    Shall see $P_{\mathcal{H}^*}$ (1st max) is in $\cap\mathcal{F}$.

    <u>NTS</u>: $\forall\, C \in \mathcal{F}\setminus\mathcal{H}^*$, $P_{\mathcal{H}^*} \in C$

    By hypothesis, $\underbrace{\mathcal{H}^* \cup \{C\}}_{\le(d+1)\text{-tuple}} \in \mathcal{I}(\mathcal{F})$

- By Lem (i), $P_{\mathcal{H}^* \cup \{C\}} = P_{\mathcal{H}^*} \Rightarrow P_{\mathcal{H}^*} \in C$. ▨