Lecture 3

Recap

Carathéodory

Radon $\longrightarrow$ Helly

every $(d+1)$-tuple intersecting
$\Rightarrow$ all intersecting

Rmk: Radon $\Rightarrow$ Helly, no geometry involved
works also in abstract setting.

§ Radon $\Rightarrow$ Carathéodory $(a \in \operatorname{conv} X)$

Pf:
- Take minimal set $\{x_1, \dots, x_m\} \subseteq X$ s.t.
  $a \in \operatorname{conv}\{x_1, \dots, x_m\}$, may assume $m \geq d+2$

$(*) \cdots \Rightarrow a = \sum_{i=1}^{m} c_i x_i$, $\sum c_i = 1$, $c_i \geq 0$

- Radon $\Rightarrow$ $\{x_1, \dots, x_m\} = A_1 \cup A_2$ s.t. $\operatorname{conv} A_1 \cap \operatorname{conv} A_2 \neq \phi$

  Let $y \in \operatorname{conv} A_1 \cap \operatorname{conv} A_2$

  $$y = \sum_{x_i \in A_1} \alpha_i x_i = \sum_{x_i \in A_2} \beta_i x_i, \quad \sum \alpha_i = \sum \beta_i = 1, \ \alpha_i, \beta_i \geq 0$$

"Rewrite 0": $\forall t \in \mathbb{R}$ $\Rightarrow 0 = t\left(\sum_{A_1} \alpha_i x_i - \sum_{A_2} \beta_i x_i\right)$

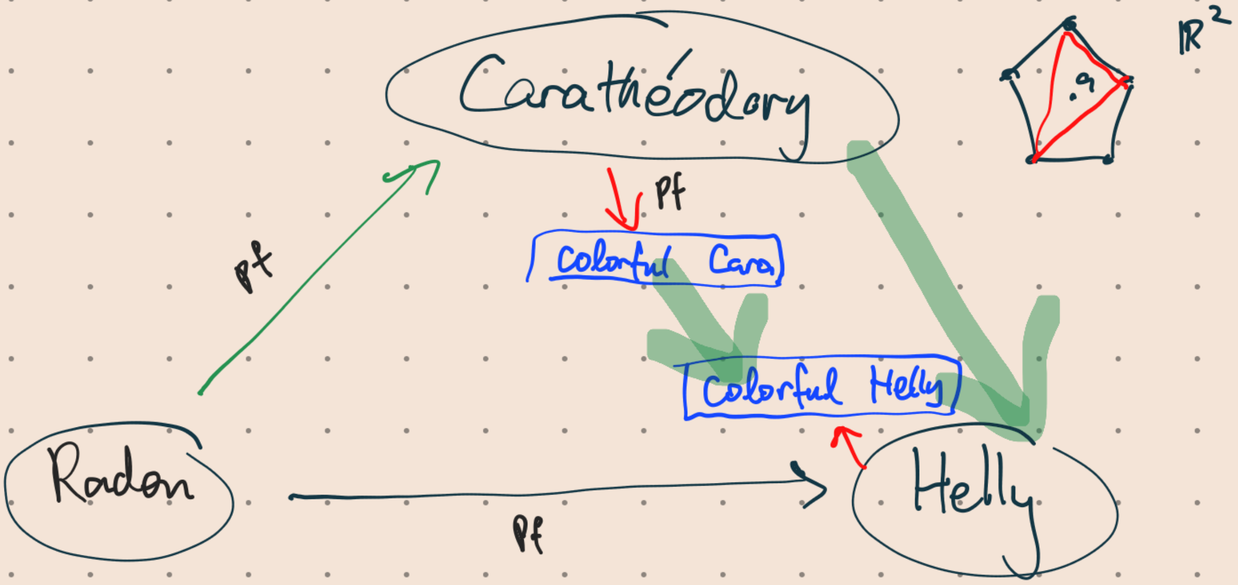
- add 0 to $(*)$

$$\Rightarrow a = \sum_{A_1} \underline{(c_i + t\alpha_i)} x_i + \sum_{A_2} (c_i - t\beta_i) x_i$$

Choose $t \geq 0$ so that $\min\{c_i - t\beta_i\} = 0$

Need to make sure $1 = \sum_{x_i \in A_1} (c_i + t\alpha_i) + \sum_{A_2} (c_i - t\beta_i)$

↯ minimality of $\{x_1, \dots, x_m\}$ ∎



Colorful Carathéodory [Bárány 82]

Let $P_1, \dots, P_{d+1} \subseteq \mathbb{R}^d$ be finite pt sets in $\mathbb{R}^d$

w./ $0 \in \operatorname{conv} P_i$, $\forall i \in [d+1]$

$\Longrightarrow$ $\exists$ transversal (rainbow) $v_1 \in P_1, \dots, v_{d+1} \in P_{d+1}$ s.t.

$$0 \in \operatorname{conv} \{v_1, \dots, v_{d+1}\}$$

Rmk. Take $P = P_1 = \dots = P_{d+1} \Longrightarrow$ Carathéodory

Pf: (Cara. $\Rightarrow$ colorful Cara.)

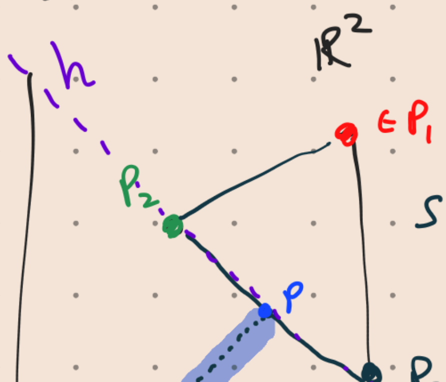
$S \subseteq P_1 \times \cdots \times P_{d+1}$

- Take rainbow d, simplex conv S closest to 0

and let $p^{\in S}$ be the closest pt

- Let h be the hyperplane $\perp$ seg OP.

Note that S is on the other side of h.



• $p \in \operatorname{conv} S \cap h = \operatorname{conv}(\underline{S \cap h})$

dim $(d-1)$

• Carathéodory applied to $S \cap h$ (dim $d-1$)

$\Rightarrow$ $p \in$ conv hull of d pts in $S \cap h$

(colors) $\rightarrow S'$

• Say P_1 is ~~the~~ a missing color. (showing up in $S \setminus S'$)

$\Rightarrow$ P_1 should contains a pt q on diff. side of h than S

o.w. $\operatorname{conv} P_1 \not\ni 0$

• Take $S' \cup \{q\}$ rainbow $(d+1)$-tuple

closer to 0 than S $\lightning$. $\blacksquare$

• <u>Colorful cone Carathéodory</u>

$A_1, \dots, A_d \subseteq \mathbb{R}^d$, $a \in \bigcap_{i \in [d]} \operatorname{cone} A_i$

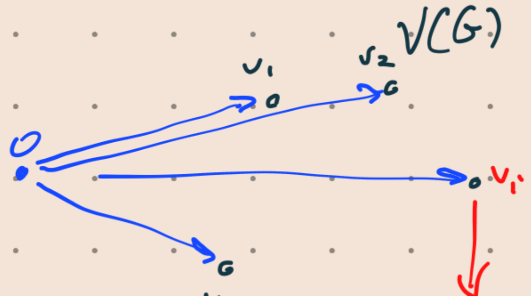
$\Rightarrow$ $\exists$ transversal $a_i \in A_i$, $i \in [d]$ s.t. $a \in \operatorname{cone}\{a_1, \dots, a_d\}$.

§ Application of colorful Cara. to directed cycles.

<u>Thm</u> [Lovász] • G n-vx directed graph

• n directed cycles $C_1, \dots, C_n$ in G

$\Rightarrow$ $\exists$ arcs $a_i \in C_i$ s.t. $\{a_1, \dots, a_n\}$ contain a directed cycle.



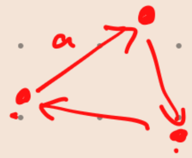
<u>Pf</u>: • Correspond vxs $v_1, \dots, v_n$ of G to standard basis

$e_1, \dots, e_n \in \mathbb{R}^n$ v_n v_j

arc $a = \overrightarrow{v_i v_j} \longleftrightarrow$ pt $e_j - e_i = p_a$

- Note that all p_a corresponding to arcs $a \in E(G)$ lie in the hyperplane $\sum_{i=1}^{n} x_i = 0$ dim $n-1$

- $\forall$ cycle C_i, $i \in [n]$: $\sum_{a \in C_i} p_a = 0$



$\Rightarrow 0 \in \operatorname{conv}\{p_a : a \in C_i\}$

- n colors $(C_1, \dots, C_n)$ dim $n-1$

colorful Cara. $\Rightarrow$ $\exists\, a_i \in C_i \ \forall i \in [n]$ s.t.

$0 \in \operatorname{conv}\{p_{a_1}, \dots, p_{a_n}\}$

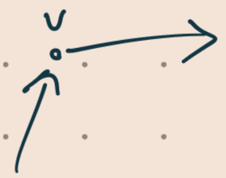
- Take minimal $I \subseteq [n]$: $(*)$ $0 = \sum_{i \in I} \alpha_i p_{a_i}$, $\alpha_i > 0$ ← minimality

D

Consider the sub-directed graph with arcs a_i, $i \in I$

$D = \bigcup_{i \in I} a_i$ contains a cycle

- Note: $\forall$ vertex $v \in D$ $d^+(v), d^-(v) \geq 1$



as otherwise the coordinate corresponding to $v \neq 0$ in $(*)$

$\Rightarrow D$ contain a cycle

minimality of $I \Rightarrow D =$ cycle. ∎

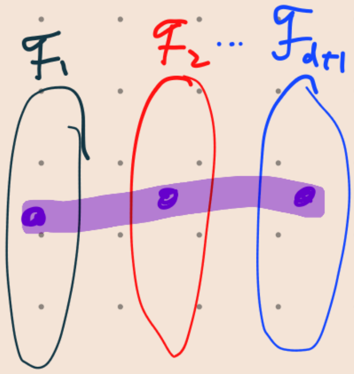
§ Colorful Helly

Thm $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$ s.t. every transversal is intersecting

i.e. $\forall\, C_1 \in \mathcal{F}_1, \dots, C_{d+1} \in \mathcal{F}_{d+1} \Rightarrow \bigcap_{i \in [d+1]} C_i \neq \emptyset$

$\Longrightarrow \exists$ (color) $i \in [d+1]$

s.t. $\underbrace{\bigcap \mathcal{F}_i}_{= \bigcap_{C \in \mathcal{F}_i} C} \neq \emptyset$



Rmk: Take $\mathcal{F} = \mathcal{F}_1 = \cdots = \mathcal{F}_{d+1} \Rightarrow$ Helly.
all colors the same