Lecture 2

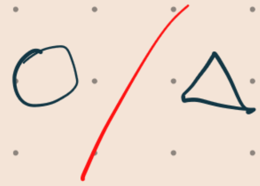
Recap:

- $\operatorname{aff} X = \{\sum \alpha_i x_i : \sum \alpha_i = 1\}$
- $\operatorname{cone} X = \{\sum \alpha_i x_i : \alpha_i \geq 0 \ \forall i\}$
- $\operatorname{conv} X = \{\sum \alpha_i x_i : \sum \alpha_i = 1, \ \alpha_i \geq 0 \ \forall i\}$

$X = \{x_1, \dots, x_n\} \subseteq \mathbb{R}^d$ aff indep $\iff$ $\binom{x_i}{1} \in \mathbb{R}^{d+1}$ linear indep.

Fact: In $\mathbb{R}^d$, max # aff. indep pts is $d+1$.

- separation thm:



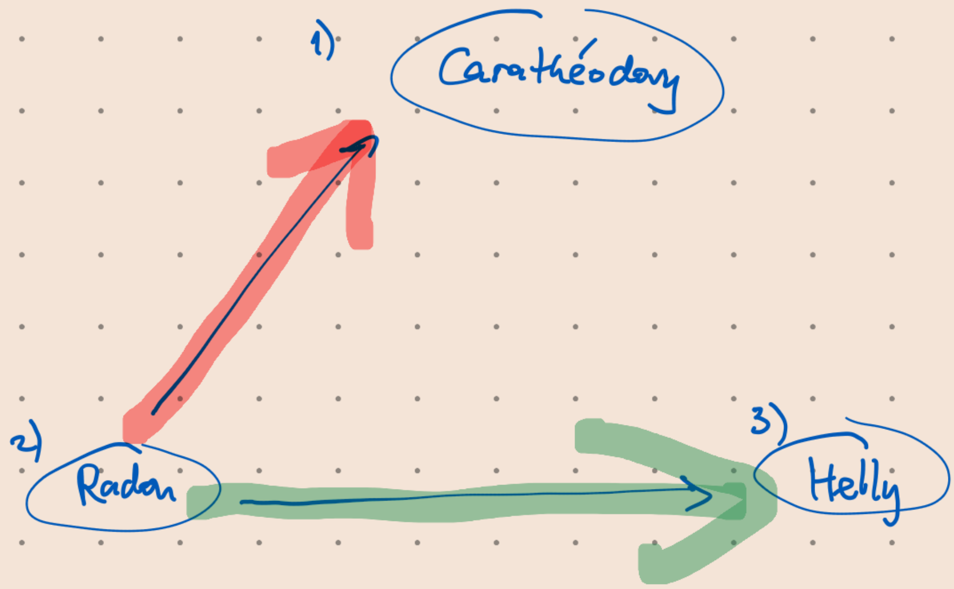
- Farkas lem.: either $\exists\, x \geq 0$ s.t. $Ax = b$

  or $\exists$ obstruction y s.t. $\begin{cases} A^T y \geq 0 \\ b^T y < 0 \end{cases}$

- Geom. version:

  closed half-spaces $a_i \cdot x \leq b_i$ no common pt $\Rightarrow$ $\binom{0}{1} \in \operatorname{cone}\left\{\binom{a_i}{b_i}\right\}$

Plan



§ Carathéodory

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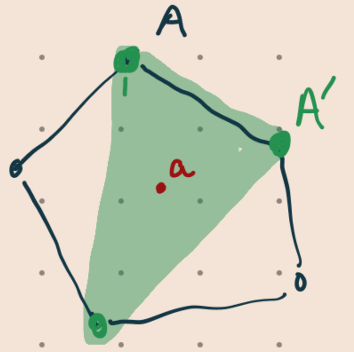
Thm [↓] $\forall A \subseteq \mathbb{R}^d$, $\forall a \in \operatorname{conv} A$ $\Rightarrow$ $\exists A' \subseteq A$ s.t.
- $|A'| \le d+1$
- $a \in \operatorname{conv} A'$

Ex $d=2$



Rmk. 1) $\operatorname{conv} A$ is covered by A-simplices.

Def: Simplex conv hull of a set of aff. indep pts.

2) We can use it to test whether $X = \{x_1, \dots, x_n\} \subseteq \mathbb{R}^2$ in convex position:

Carathéodory $\Rightarrow$ Enough to check all 4 of them (if all yes $\Rightarrow$ X in conv. position)

Pf (using aff. dep.) $A = \{x_1, \dots, x_n\}$

- Use induction on $n = |A| \geq d+2$
- Take a 'minimal' conv. combination of pts in A containing a.

i.e. $\boxed{a = \sum_{i=1}^m \alpha_i x_i}$,
- $\sum \alpha_i = 1$
- $\alpha_i \geq 0$
- m minimum $\geq d+2$

As $m \geq d+2$
- $\Rightarrow x_1, \dots, x_m$ are aff. dep. i.e

$$\begin{cases} \boxed{\sum_{i=1}^m \beta_i x_i = 0} \leftarrow \text{useful!} \\ \sum_{i=1}^m \beta_i = 0 \end{cases}$$
↑ not all 0.

$\Box - t \cdot \Box$

$\Rightarrow \forall t \in \mathbb{R}$, $a = \sum_{i=1}^m (\alpha_i - t\beta_i) x_i$

Choose t s.t. Need $\begin{cases} \alpha_i - t\beta_i \geq 0 & (*) \\ \text{one of them} = 0 \end{cases}$ ≥ 0

Try $t > 0$ $\xrightarrow{(*)}$ $\begin{cases} \beta_i < 0 \Rightarrow \alpha_i - t\beta_i > 0 \\ \beta_i > 0 \Rightarrow \dfrac{\alpha_i}{\beta_i} \geq t \end{cases}$ ☺

- Suggest $\Rightarrow$ $t = \min\limits_{i \in [m]} \left\{ \dfrac{\alpha_i}{\beta_i} : \beta_i > 0 \right\}$. ▨

Thm [Cone version of Carathéodory]

$\forall A \subseteq \mathbb{R}^d$, $\forall a \in \operatorname{cone} A$ $\Longrightarrow$ $\exists A' \subseteq A$ s.t.
- $|A'| \leq d$
- $a \in \operatorname{cone} A'$.

- Pf uses instead linear dependency.

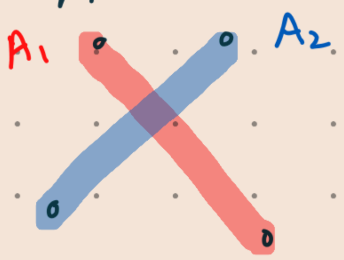
§ Radon.

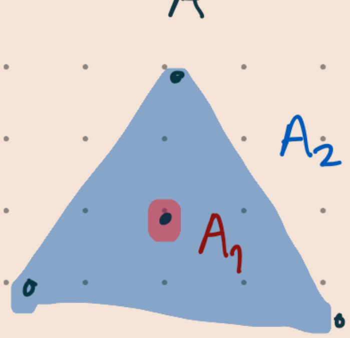
Lem [↙] $\forall A \subseteq \mathbb{R}^d$, $|A| = d+2$ pts

$\Longrightarrow \exists A = A_1 \cup A_2$ s.t. $\operatorname{conv} A_1 \cap \operatorname{conv} A_2 \neq \emptyset$.

↖ Radon partition ↖ Radon pt.

Ex $d=2$





Rmk • $d+2$ here is optimal. Consider a simplex Δ^d

$d=2$  $d=3$ 

Pf: • $|A| = d+2 \Rightarrow$ aff. dep.

$$\begin{cases} \sum_{i=1}^{d+2} \alpha_i x_i = 0 \cdots (1) \\ \sum_{i=1}^{d+2} \alpha_i = 0 \cdots (2) \end{cases}$$

(partition x_i by posi. coeff. / neg. coeff.)

For each i : $\alpha_i = \begin{cases} \beta_i & \text{if } \alpha_i \geq 0 \\ -\gamma_i & \text{if } \alpha_i < 0 \end{cases}$

$(2) \Rightarrow \sum \beta_i = \sum \gamma_i =: T$

$(1) \Rightarrow \underbrace{\frac{1}{T}\sum \beta_i x_i}_{\oplus} = \underbrace{\frac{1}{T}\sum \gamma_i x_i}_{\ominus} \Leftarrow$ scale by T to get Radon pt

§ Helly

<u>Thm</u> [Helly]. $\mathcal{F} = \{C_1, \dots, C_n\}$ collection of convex sets in $\mathbb{R}^d$, $n \geq d+1$

if all $\binom{\mathcal{F}}{d+1}$ are intersecting $\Rightarrow$ $\mathcal{F}$ is intersecting $\cap \mathcal{F} \neq \emptyset$.

i.e. $C_1 \cap C_2 \cap \cdots \cap C_{d+1} \neq \emptyset$

<u>Ex</u> $d = 1$ if every two interval intersects

$\mathcal{F} = \{I_1, \dots, I_n\}$ $\Rightarrow$ $\exists$ one pts in all interval.

<u>Pf</u> (Helly 1-dim)

- Let x be the 1^{st} max. (right endpt.) of an interval (J) in $\mathcal{F}$

  Shall see $x \in \cap \mathcal{F}$

  $\forall I \in \mathcal{F}$ NTS $x \in I$

- left end pt of $I \leq x \Rightarrow x \in I$
- if not $I \cap J = \phi$ ▨

§ Radon ⇒ Helly

$\mathcal{F} = \{C_1, \ldots, C_n\}$

<u>Pf</u>

- By induction on n, suffices to show $n = d+2$ case.
- $\overset{\text{hypothesis}}{\Longrightarrow} \forall i \in [d+2], \ \exists x_i \in \bigcap_{[d+2] \setminus \{i\}} C_j$

$A = \{x_1, \ldots, x_{d+2}\}$

- $\overset{\text{Radon}}{\Longrightarrow} \exists A_1 \cup A_2 = A$ s.t. $\operatorname{conv} A_1 \cap \operatorname{conv} A_2 \neq \phi$
  $x \in$

  Take $x \in \operatorname{conv} A_1 \cap \operatorname{conv} A_2$
  Shall see $x \in \cap \mathcal{F}$

- Fix $C_i \in \mathcal{F}$, say $i \notin A_1$

  $\Rightarrow$ all x_j, $j \in A_1$, lie in C_i

  $A_1 \subseteq C_i \Rightarrow x \in C_i \quad \forall i \notin A_1$ ▨